Name: ____________________________________ Date: __________________

INTEGER EXPONENTS
COMMON CORE ALGEBRA II

We just finished our review of linear functions. Linear functions are those that grow by equal differences for equal intervals. In this unit we will concentrate on exponential functions which grow by equal factors for equal intervals. To understand exponential functions, we first need to understand exponents.

***Exercise* #1:** The following sequence shows powers of 3 by repeatedly multiplying by 3. Fill in the missing blanks.

3^{-3}	3^{-2}	3^{-1}	3^{0}	3^{1}	3^{2}	3^{3}
________	________	________	________	3	9	________

×3 ×3 ×3 ×3 ×3 ×3

This pattern can be duplicated for any **base** raised to any **integer exponent**. Because of this we can now define positive, negative, and zero exponents in terms of multiplying the number 1 repeatedly or dividing the number 1 repeatedly.

INTEGER EXPONENT DEFINITIONS

If n is any positive integer then:

1. $b^n = 1 \cdot \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b \cdot b}_{n\text{-times}}$

2. $b^0 = 1$

3. $b^{-n} = \dfrac{1}{\underbrace{b \cdot b \cdot b \cdot \ldots \cdot b \cdot b}_{n\text{-times}}} = \dfrac{1}{b^n}$

***Exercise* #2:** Given the exponential function $f(x) = 20(2)^x$ evaluate each of the following without using your calculator. Show the calculations that lead to your final answer.

(a) $f(2)$

(b) $f(0)$

(c) $f(-2)$

(d) When x increases by 3, by what factor does y increase? Explain your answer.

There are many basic **exponent properties or laws** that are critically important and that can be investigated using integer exponent examples. Two of the very important ones we will see next.

***Exercise* #3:** For each of the following, write the product as a single exponential expression. Write (a) and (b) as extended products first (if necessary).

(a) $2^3 \cdot 2^4$

(b) $2^6 \cdot 2^2$

(c) $2^m \cdot 2^n$

It's clear why the exponent law that you generalized in part (c) works for positive integer exponents. But, does it also make sense within the context of our negative exponents.

***Exercise* #4:** Consider now the product $2^3 \cdot 2^{-1}$.

(a) Use the exponent law found in Exercise 3(c) to write this as a single exponential expression.

(b) Evaluate $2^3 \cdot 2^{-1}$ by first rewriting 2^3 and 2^{-1} and then simplifying.

(c) Do your answers from (a) and (b) support the extension of the **Addition Property of Exponents** to negative powers as well? Explain.

Let's look at another important exponent property.

***Exercise* #5:** For each of the following, write the exponential expression in the form 3^x. Write (a) and (b) as extended products first (if necessary).

(a) $\left(3^2\right)^3$

(b) $\left(3^4\right)^2$

(c) $\left(3^m\right)^n$

Again, let's look at how the **Product Property of Exponents** still holds for negative exponents.

***Exercise* #6:** Consider the expression $\left(3^{-2}\right)^4$. Show this expression is equivalent to 3^{-8} by first rewriting 3^{-2} in fraction form.

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INTEGER EXPONENTS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Write each of the following exponential expressions without the use of exponents such as we did in lesson *Exercise* #1.

(a)

2^{-3}	______	
		$\times 2$
2^{-2}	______	
		$\times 2$
2^{-1}	______	
		$\times 2$
2^{0}	______	
		$\times 2$
2^{1}	2	
		$\times 2$
2^{2}	4	
		$\times 2$
2^{3}	______	

(b)

5^{-3}	______	
		$\times 5$
5^{-2}	______	
		$\times 5$
5^{-1}	______	
		$\times 5$
5^{0}	______	
		$\times 5$
5^{1}	5	
		$\times 5$
5^{2}	25	
		$\times 5$
5^{3}	______	

2. Now let's go the other way around. For each of the following, determine the integer value of n that satisfies the equation. The first is done for you.

(a) $2^n = \frac{1}{8}$

$2^n = \frac{1}{2^3}$

$2^n = 2^{-3}$

$n = -3$

(b) $4^n = 16$

(c) $3^n = \frac{1}{81}$

(d) $7^n = 1$

(e) $5^n = \frac{1}{25}$

(f) $10^n = \frac{1}{10,000}$

(g) $13^n = 1$

(h) $2^n = \frac{1}{32}$

3. Use the **Addition Property of Exponents** to simplify each expression. Then, find a final numerical answer *without* using your calculator.

(a) $2^{-5}\cdot 2^{3}\cdot 2^{4}$ (b) $5^{3}\cdot 5^{7}\cdot 5^{-10}$ (c) $10^{3}\cdot 10^{-7}\cdot 10^{2}$

4. Use the **Product Property of Exponents** to simplify each exponential expression. You do not need to find a final numerical answer.

(a) $\left(2^{3}\right)^{4}$ (b) $\left(3^{-2}\right)^{2}$ (c) $\left(\left(5^{2}\right)^{-4}\right)^{-2}$

5. The exponential expression $\left(\frac{1}{8}\right)^{4}$ is equivalent to which of the following? Explain your choice.

(1) 4^{-8} (3) 8^{-2}

(2) 2^{-12} (4) 32^{-1}

____

REASONING

6. How can you use the fact that $25^{2}=625$ to show that $5^{-4}=\frac{1}{625}$? Explain your process of thinking.

7. We've extended the two fundamental exponent properties to negative as well as positive integers. What would happen if we extended the **Product Exponent Property** to a fractional exponent like $\frac{1}{2}$? Let's play around with that idea.

(a) Use the **Product Property of Exponents** to justify that $\left(9^{\frac{1}{2}}\right)^{2}=9$.

(b) What other number can you square that results in 9? Hmm...

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RATIONAL EXPONENTS
COMMON CORE ALGEBRA II

When you first learned about exponents, they were always **positive integers**, and just represented **repeated multiplication**. And then we had to go and introduce **negative exponents**, which really just represent **repeated division**. Today we will introduce **rational (or fractional) exponents** and extend your exponential knowledge that much further.

***Exercise* #1:** Recall the **Product Property of Exponents** and use it to rewrite each of the following as a simplified exponential expression. There is no need to find a final numerical value.

(a) $\left(2^3\right)^4$ (b) $\left(5^{-2}\right)^5$ (c) $\left(3^7\right)^0$ (d) $\left(\left(4^2\right)^{-2}\right)^2$

We will now use the Product Property to extend our understanding of exponents to include **unit fraction** exponents (those of the form $1/n$ where n is a positive integer).

***Exercise* #2:** Consider the expression $16^{1/2}$.

(a) Apply the Product Property to simplify $\left(16^{1/2}\right)^2$. What other number squared yields 16?

(b) You can now say that $16^{1/2}$ is equivalent to what more familiar quantity?

This is remarkable! An exponent of $1/2$ is equivalent to a square root of a number!!!

***Exercise* #3:** Test the equivalence of the $1/2$ exponent to the square root by using your calculator to evaluate each of the following. Be careful in how you enter each expression.

(a) $25^{1/2} =$ (b) $81^{1/2} =$ (c) $100^{1/2}$

We can extend this now to all levels of roots, that is square roots, cubic roots, fourth roots, etcetera.

UNIT FRACTION EXPONENTS

For n given as a positive integer: $b^{1/n} = \sqrt[n]{b}$

***Exercise* #4:** Rewrite each of the following using roots instead of fractional exponents. Then, if necessary, evaluate using your calculator to guess and check to find the roots (don't use the generic root function). Check with your calculator.

(a) $125^{1/3}$ (b) $16^{1/4}$ (c) $9^{-1/2}$ (d) $32^{-1/5}$

We can now combine traditional integer powers with unit fractions in order interpret any exponent that is a **rational number**, i.e. the **ratio of two integers**. The next exercise will illustrate the thinking. Remember, we want our exponent properties to be consistent with the structure of the expression.

***Exercise* #5:** Let's think about the expression $4^{3/2}$.

(a) Fill in the missing blank and then evaluate this expression:

$$4^{3/2} = (\qquad)^{1/2}$$

(b) Fill in the missing blank and then evaluate this expression:

$$4^{3/2} = (\qquad)^{3}$$

(c) Verify both (a) and (b) using your calculator.

(d) Evaluate $27^{2/3}$ without your calculator. Show your thinking. Verify with your calculator.

RATIONAL EXPONENT CONNECTION TO ROOTS

For the rational number m/n we define $b^{m/n}$ to be: $\sqrt[n]{b^m}$ or $\left(\sqrt[n]{b}\right)^m$.

***Exercise* #6:** Evaluate each of the following exponential expressions involving rational exponents without the use of your calculator. Show your work. Then, check your final answers with the calculator.

(a) $16^{3/4}$ (b) $25^{3/2}$ (c) $8^{-2/3}$

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RATIONAL EXPONENTS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Rewrite the following as equivalent roots and then evaluate as many as possible **without your calculator.**

(a) $36^{1/2}$ (b) $27^{1/3}$ (c) $32^{1/5}$ (d) $100^{-1/2}$

(e) $625^{1/4}$ (f) $49^{1/2}$ (g) $81^{-1/4}$ (h) $343^{1/3}$

2. Evaluate each of the following by considering the root and power indicated by the exponent. Do as many as possible **without your calculator**.

(a) $8^{2/3}$ (b) $4^{3/2}$ (c) $16^{-3/4}$ (d) $81^{5/4}$

(e) $4^{-5/2}$ (f) $128^{3/7}$ (g) $625^{3/4}$ (h) $243^{3/5}$

3. Given the function $f(x)=5(x+4)^{3/2}$, which of the following represents its *y*-intercept?

(1) 40 (3) 4

(2) 20 (4) 30

____

4. Which of the following is equivalent to $x^{-1/2}$?

(1) $-\frac{1}{2}x$　　(3) $\frac{1}{\sqrt{x}}$

(2) $-\sqrt{x}$　　(4) $-\frac{1}{2x}$　　__________

5. Written without fractional or negative exponents, $x^{-3/2}$ is equal to

(1) $-\frac{3x}{2}$　　(3) $\frac{1}{\sqrt{x^3}}$

(2) $\frac{1}{\sqrt[3]{x^2}}$　　(4) $-\frac{1}{\sqrt{x}}$　　__________

6. Which of the following is *not* equivalent to $16^{3/2}$?

(1) $\sqrt{4096}$　　(3) 4^3

(2) 64　　(4) $\sqrt{16^3}$　　__________

Reasoning

7. Marlene claims that the square root of a cube root is a sixth root? Is she correct? To start, try rewriting the expression below in terms of fractional exponents. Then apply the **Product Property of Exponents**.

$\sqrt{\sqrt[3]{a}}$

8. We should know that $\sqrt[3]{8}=2$. To see how this is equivalent to $8^{1/3}=2$ we can solve the equation $8^n=2$. To do this, we can rewrite the equation as:

$$\left(2^3\right)^n=2^1$$

How can we now use this equation to see that $8^{1/3}=2$?

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EXPONENTIAL FUNCTION BASICS
COMMON CORE ALGEBRA II

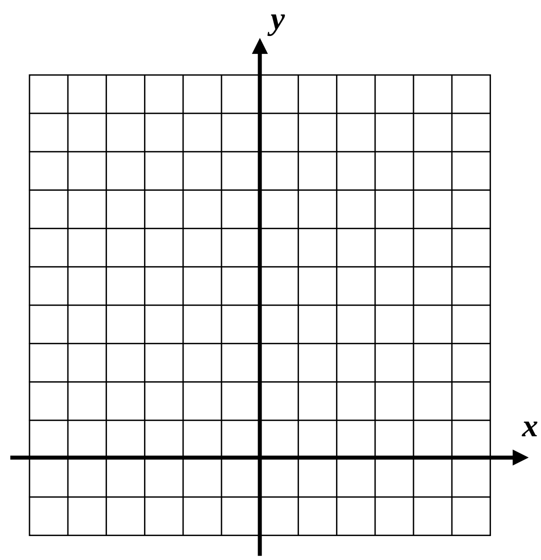
You studied exponential functions extensively in Common Core Algebra I. Today's lesson will review many of the basic components of their graphs and behavior. Exponential functions, those whose exponents are variable, are extremely important in mathematics, science, and engineering.

BASIC EXPONENTIAL FUNCTIONS

$y = b^x$ where $b > 0$ and $b \neq 1$

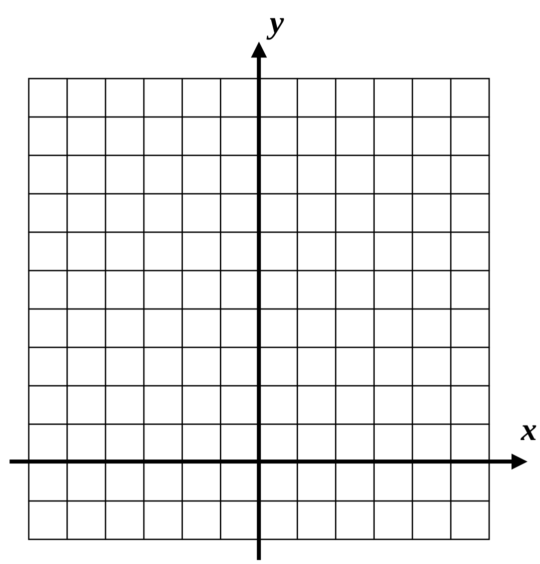
***Exercise* #1:** Consider the function $y = 2^x$. Fill in the table below without using your calculator and then sketch the graph on the grid provided.

x	$y = 2^x$
−3	
−2	
−1	
0	
1	
2	
3	



***Exercise* #2:** Now consider the function $y = \left(\frac{1}{2}\right)^x$. Using your calculator to help you, fill out the table below and sketch the graph on the axes provided.

x	$y = \left(\frac{1}{2}\right)^x$
−3	
−2	
−1	
0	
1	
2	
3	

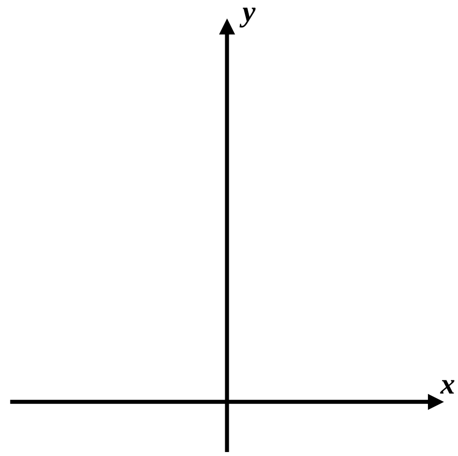


***Exercise* #3:** Based on the graphs and behavior you saw in *Exercises* #1 and #2, state the domain and range for an exponential function of the form $y = b^x$.

Domain (input set): Range (output set):

***Exercise* #4:** Are exponential functions one-to-one? How can you tell? What does this tell you about their inverses?

***Exercise* #5:** Now consider the function $y = 7(3)^x$.

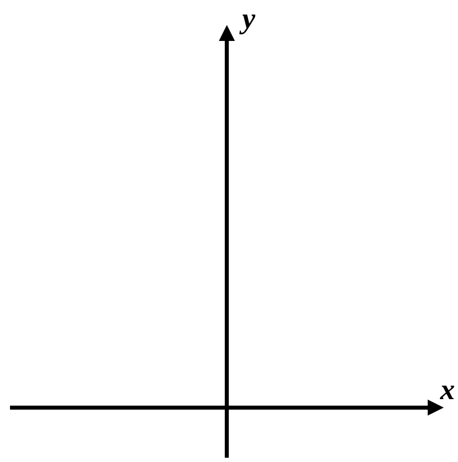


(a) Determine the *y*-intercept of this function algebraically. Justify your answer.

(b) Does the exponential function increase or decrease? Explain your choice.

(c) Create a rough sketch of this function, labeling its *y*-intercept.

***Exercise* #6:** Consider the function $y = \left(\frac{1}{3}\right)^x + 4$.



(a) How does this function's graph compare to that of $y = \left(\frac{1}{3}\right)^x$? What does adding 4 do to a function's graph?

(b) Determine this graph's *y*-intercept algebraically. Justify your answer.

(c) Create a rough sketch of this function, labeling its *y*-intercept.

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EXPONENTIAL FUNCTION BASICS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following represents an exponential function?

(1) $y = 3x - 7$ (3) $y = 3(7)^x$

(2) $y = 7x^3$ (4) $y = 3x^2 + 7$ __________

2. If $f(x) = 6(9)^x$ then $f\left(\frac{1}{2}\right) = ?$ (Remember what we just learned about fractional exponents and do withou a calculator.)

(1) $\frac{7}{2}$ (3) 27

(2) 18 (4) $\frac{15}{2}$ __________

3. If $h(x) = 3^x$ and $g(x) = 5x - 7$ then $h(g(2)) =$

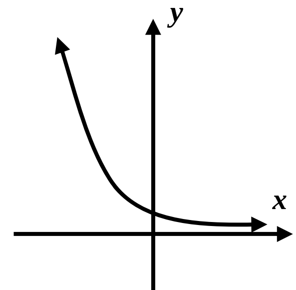
(1) 18 (3) 38

(2) 12 (4) 27 __________

4. Which of the following equations could describe the graph shown below?

(1) $y = x^2 + 1$ (3) $y = -2x + 1$

(2) $y = \left(\frac{2}{3}\right)^x$ (4) $y = 4^x$



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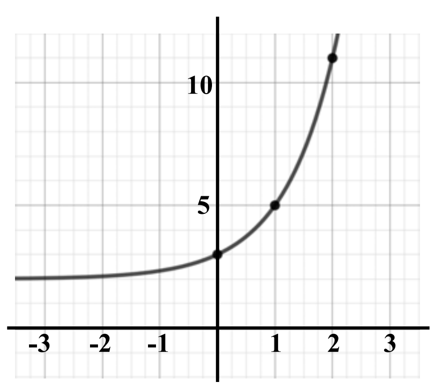
5. Which of the following equations represents the graph shown?

(1) $y = 5^x$ (3) $y = \left(\frac{1}{2}\right)^x + 2$

(2) $y = 4^x + 1$ (4) $y = 3^x + 2$



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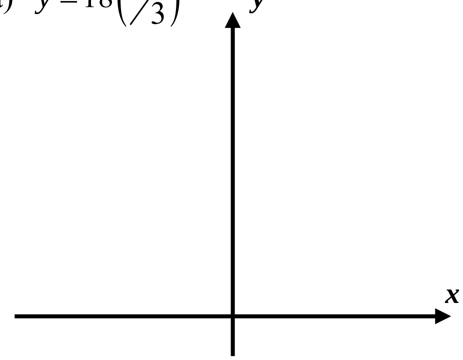
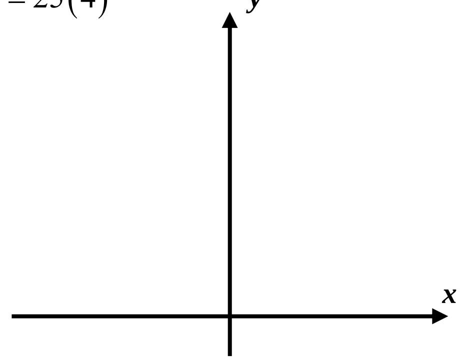
6. Sketch graphs of the equations shown below on the axes given. Label the *y*-intercepts of each graph.

(a) $y=18\left(\frac{1}{3}\right)^x$



(b) $y=25(4)^x$



APPLICATION

7. The Fahrenheit temperature of a cup of coffee, *F*, starts at a temperature of 185°F. It cools down according to the exponential function $F(m)=113\left(\dfrac{1}{2}\right)^{m/20}+72$, where *m* is the number minutes it has been cooling.

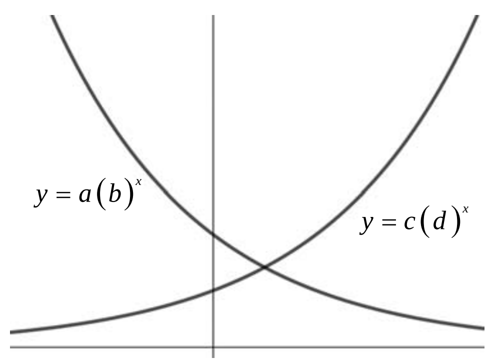
(a) How do you interpret the statement that $F(60)=85$?

(b) Determine the temperature of the coffee after one day using your calculator. What do you think this temperature represents about the physical situation?

REASONING

8. The graph below shows two exponential functions, with real number constants *a*, *b*, *c*, and *d*. Given the graphs, only one pair of the constants shown below could be equal in value. Determine which pair could be equal and explain your reasoning.

b and *d* *a* and *b* *a* and *c*



9. Explain why the equation below can have no real solutions. If you need to, graph both sides of the equation using your calculator to visualize the reason.

$$3^x+5=2$$

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WRITING EQUATIONS OF EXPONENTIAL FUNCTIONS
COMMON CORE ALGEBRA II

One of the skills that you acquired in Common Core Algebra I was the ability to write equations of exponential functions if you had information about the starting value and base (multiplier or growth constant). Let's review a very basic problem.

***Exercise* #1:** An exponential function of the form $f(x)=a(b)^x$ is presented in the table below. Determine the values of *a* and *b* and explain your reasoning.

x	0	1	2	3
$f(x)$	5	15	45	135

$a=$ _________

$b=$ _________

Final Equation: _____________________ Explanation:

Finding an exponential equation becomes much more challenging if we do not have output values for inputs that are increasing by unit values (increasing by 1 unit at a time). Let's start with a basic problem.

***Exercise* #2:** For an exponential function of the form $f(x)=a(b)^x$, it is known that $f(0)=8$ and $f(3)=1000$.

(a) Use the fact that $f(0)=8$ to determine the value of *a*. Show your thinking.

(b) Use you answer from part (a) and the fact that $f(3)=1000$ to set up an equation to solve for *b*. You will solve for *b* in part (c).

(c) Solve for the value of *b* using properties of exponents.

(d) Determine the value of $f(2)$

***Exercise* #3:** An exponential function exists such that $f(4)=3$ and $f(6)=48$, which of the following must be the value of its base? Explain or illustrate your thinking.

(1) $b=16$

(2) $b=2$

(3) $b=6$

(4) $b=4$

____

Now, let's work with the most generic type of problem. Just like with lines, **any two (non-vertically aligned) points will uniquely determine the equation of an exponential function**.

***Exercise* #4:** An exponential function of the form $y = a(b)^x$ passes through the points $(2, 36)$ and $(5, 121.5)$.

(a) By substituting these two points into the general form of the exponential, create a **system of equations** in the constants *a* and *b*.

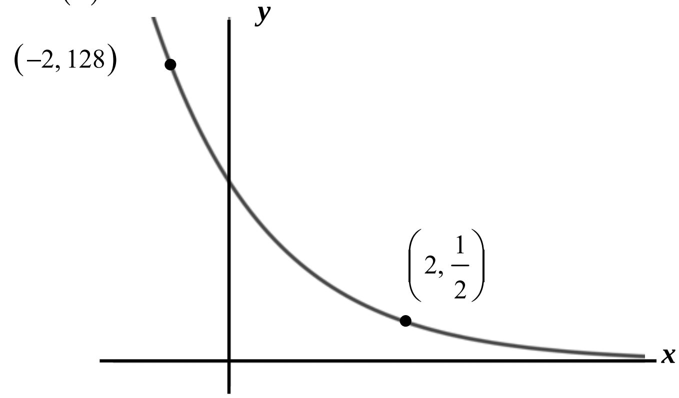
(b) Divide these two equations to eliminate the constant *a*. Recall that when dividing to like bases, you subtract their exponents.

(c) Solve the resulting equation from (b) for the base, *b*.

(d) Use your value from (c) to determine the value of *a*. State the final equation.

Let's now get some practice on this with a decreasing exponential function.

***Exercise* #5:** Find the equation of the exponential function shown graphed below. Be careful in terms of your exponent manipulation. State your final answer in the form $y = a(b)^x$.



***Exercise* #6:** A bacterial colony is growing at an exponential rate. It is known that after 4 hours, its population is at 98 bacteria and after 9 hours it is 189 bacteria. Determine an equation in $y = a(b)^x$ form that models the population, *y*, as a function of the number of hours, *x*. At what percent rate is the population growing per hour?

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FINDING EQUATIONS OF EXPONENTIAL FUNCTIONS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. For each of the following coordinate pairs, find the equation of the exponential function, in the form $y = a(b)^x$ that passes through the pair. Show the work that you use to arrive at your answer.

(a) $(0,10)$ and $(3,80)$

(b) $(0,180)$ and $(2,80)$

2. For each of the following coordinate pairs, find the equation of the exponential function, in the form $y = a(b)^x$ that passes through the pair. Show the work that you use to arrive at your answer.

(a) $(2,192)$ and $(5,12288)$

(b) $(1,192)$ and $(5,60.75)$

3. Each of the previous problems had values of a and b that were rational numbers. They do not need not be. Find the equation for an exponential function that passes through the points $(2,14)$ and $(7,205)$ in $y = a(b)^x$ form. When you find the value of b do not round your answer before you find a. Then, find both to the nearest *hundredth* and give the final equation. Check to see if the points fall on the curve.

APPLICATIONS

4. A population of koi goldfish in a pond was measured over time. In the year 2002, the population was recorded as 380 and in 2006 it was 517. Given that y is the population of fish and x is the number of years *since* 2000, do the following:

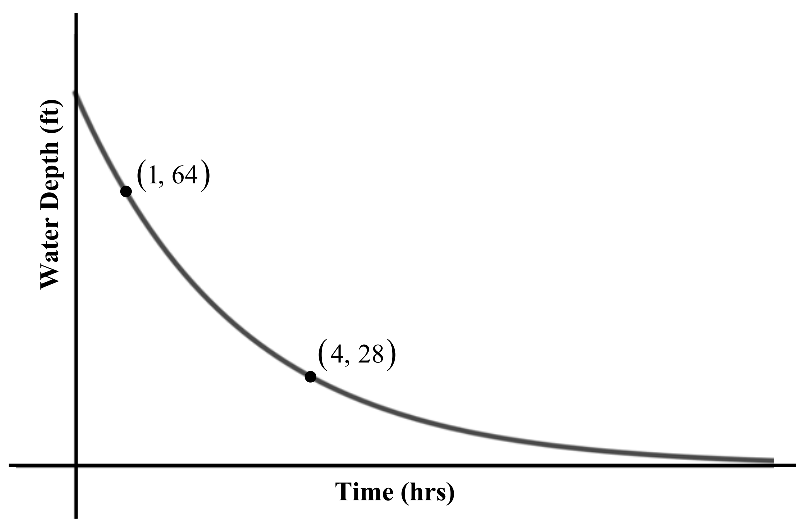
(a) Represent the information in this problem as two coordinate points.

(b) Determine a linear function in the form $y = mx + b$ that passes through these two points. Don't round the linear parameter (m and b).

(c) Determine an exponential function of the form $y = a(b)^x$ that passes through these two points. Round b to the nearest hundredth and a to the nearest *tenth*.

(d) Which model predicts a larger population of fish in the year 2000? Justify your work.

5. Engineers are draining a water reservoir until its depth is only 10 feet. The depth decreases exponentially as shown in the graph below. The engineers measure the depth after 1 hour to be 64 feet and after 4 hours to be 28 feet. Develop an exponential equation in $y = a(b)^x$ to predict the depth as a function of hours draining. Then, graph the horizontal line $y = 10$ and find its intersection to determine the time, to the nearest tenth of an hour, when the reservoir will reach a depth of 10 feet.



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THE METHOD OF COMMON BASES
COMMON CORE ALGEBRA II

There are very few algebraic techniques that **do not involve technology** to solve equations that contain **exponential expressions**. In this lesson we will look at one of the few, known as **The Method of Common Bases**.

***Exercise* #1:** Solve each of the following simple exponential equations by writing each side of the equation using **a common base**.

(a) $2^x = 16$ (b) $3^x = 27$ (c) $5^x = \frac{1}{25}$ (d) $16^x = 4$

In each of these cases, even the last, more challenging one, we could manipulate the right-hand side of the equation so that it shared a **common base** with the left-hand side of the equation. We can exploit this fact by manipulating both sides so that they have a common base. First, though, we need to review an exponent law.

***Exercise* #2:** Simplify each of the following exponential expressions.

(a) $\left(2^3\right)^x$ (b) $\left(3^2\right)^{4x}$ (c) $\left(5^{-1}\right)^{3x-7}$ (d) $\left(4^{-3}\right)^{1-x^2}$

***Exercise* #3:** Solve each of the following equations by finding a common base for each side.

(a) $8^x = 32$ (b) $9^{2x+1} = 27$ (c) $125^x = \left(\frac{1}{25}\right)^{4-x}$

***Exercise* #4:** Which of the following represents the solution set to the equation $2^{x^2-3} = 64$?

(1) $\{\pm 3\}$ (3) $\{\pm\sqrt{11}\}$

(2) $\{0, 3\}$ (4) $\{\pm\sqrt{35}\}$

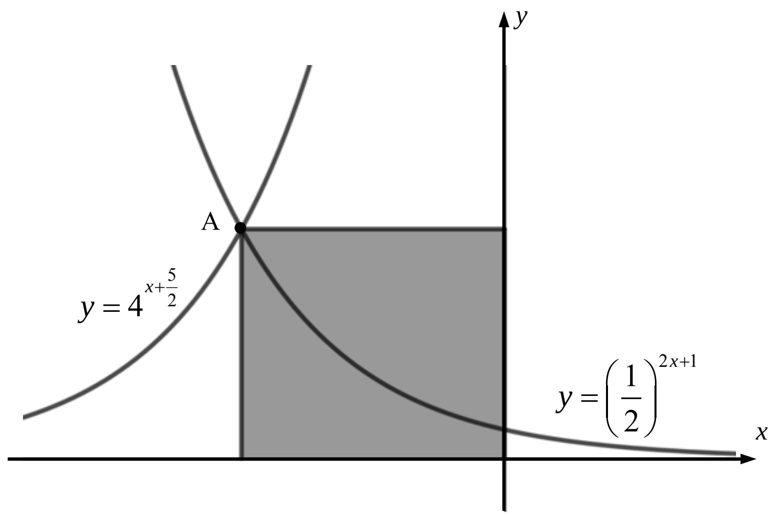
____

This technique can be used in any situation where all bases involved can be written with a common base. In a practical sense, this is rather rare. Yet, these types of algebraic manipulations help us see the **structure** in **exponential expressions**. Try to tackle the next, more challenging, problem.

Exercise #5: Two exponential curves, $y=4^{x+\frac{5}{2}}$ and $y=\left(\frac{1}{2}\right)^{2x+1}$ are shown below. They intersect at point A. A rectangle has one vertex at the origin and the other at A as shown. We want to find its area.

(a) Fundamentally, what do we need to know about a rectangle to find its area?

(b) How would knowing the coordinates of point A help us find the area?



(c) Find the area of the rectangle algebraically using the Method of Common Bases. Show your work carefully.

Exercise #6: At what *x* coordinate will the graph of $y=25^{x-a}$ intersect the graph of $y=\left(\frac{1}{125}\right)^{3x+1}$? Show the work that leads to your choice.

(1) $x=\frac{5a-1}{3}$

(2) $x=\frac{2a-3}{11}$

(3) $x=\frac{-2a+1}{5}$

(4) $x=\frac{5a+3}{2}$

____

Name: ____________________________________ Date: __________________

THE METHOD OF COMMON BASES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Solve each of the following exponential equations using the Method of Common Bases. Each equation will result in a linear equation with one solution. Check your answers.

(a) $3^{2x-5}=9$

(b) $2^{3x+7}=16$

(c) $5^{4x-5}=\frac{1}{125}$

(d) $8^{x}=4^{2x+1}$

(e) $216^{x-2}=\left(\frac{1}{1296}\right)^{3x-2}$

(f) $\left(\frac{1}{25}\right)^{x+15}=3125^{\frac{3}{5}x-1}$

2. *Algebraically* determine the intersection point of the two exponential functions shown below. Recall that most systems of equations are solved by substitution.

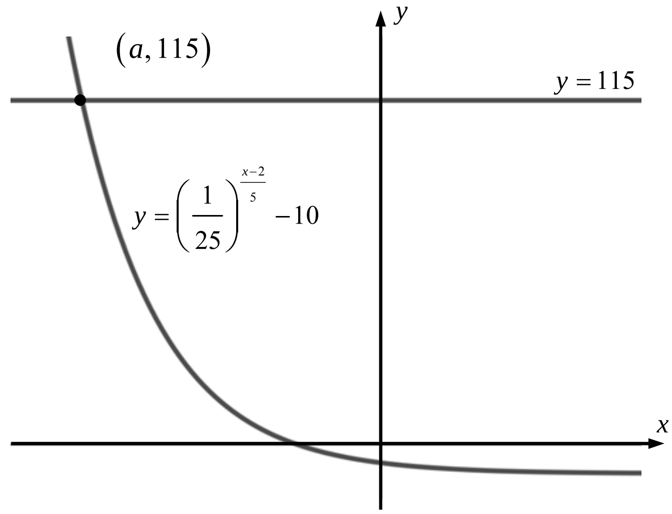
$$y=8^{x-1} \text{ and } y=4^{2x-3}$$

3. *Algebraically* determine the **zeroes** of the exponential function $f(x)=2^{2x-9}-32$. Recall that the reason it is known as a zero is because the **output is zero**.

APPLICATIONS

4. One hundred must be raised to what power in order to be equal to a million cubed? Solve this problem using the Method of Common Bases. Show the algebra you do to find your solution.

5. The exponential function $y=\left(\frac{1}{25}\right)^{\frac{x-2}{5}}-10$ is shown graphed along with the horizontal line $y=115$. Their intersection point is $(a,115)$. Use the Method of Common Bases to find the value of a. Show your work.



REASONING

6 The Method of Common Bases works because exponential functions are one-to-one, i.e. if the outputs are the same, then the inputs must also be the same. This is what allows us to say that if $2^x=2^3$, then x must be equal to 3. But it doesn't always work out so easily.

If $x^2=5^2$, can we say that x must be 5? Could it be anything else? Why does this not work out as easily as the exponential case?

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EXPONENTIAL MODELING WITH PERCENT GROWTH AND DECAY
COMMON CORE ALGEBRA II

Exponential functions are very important in modeling a variety of real world phenomena because certain things either increase or decrease by **fixed percentages** over given units of time. You looked at this in Common Core Algebra I and in this lesson we will review much of what you saw.

***Exercise* #1:** Suppose that you deposit money into a savings account that receives 5% interest per year on the amount of money that is in the account for that year. Assume that you deposit $400 into the account initially.

(a) How much will the savings account increase by over the course of the year?

(b) How much money is in the account at the end of the year?

(c) By what single number could you have multiplied the $400 by in order to calculate your answer in part (b)?

(d) Using your answer from part (c), determine the amount of money in the account after 2 and 10 years. Round all answers to the nearest cent when needed.

(e) Give an equation for the amount in the savings account $S(t)$ as a function of the number of years since the $400 was invested.

(f) Using a table on your calculator determine, to the nearest year, how long it will take for the initial investment of $400 to double. Provide evidence to support your answer.

The thinking process from *Exercise* #1 can be generalized to any situation where a quantity is increased by a fixed percentage over a fixed interval of time. This pattern is summarized below:

INCREASING EXPONENTIAL MODELS

If quantity *Q* is known to increase by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t) = Q_0(1+p)^t$$

where Q_0 represents the amount of *Q* present at $t = 0$ and *t* represents time.

***Exercise* #2:** Which of the following gives the savings *S* in an account if $250 was invested at an interest rate of 3% per year?

(1) $S = 250(4)^t$

(3) $S = (1.03)^t + 250$

(2) $S = 250(1.03)^t$

(4) $S = 250(1.3)^t$

____

Decreasing exponentials are developed in the same way, but have the percent subtracted, rather than added, to the base of 100%. Just remember, you are ultimately multiplying by the percent of the original that you will have after the time period elapses.

***Exercise* #3:** State the multiplier (base) you would need to multiply by in order to decrease a quantity by the given percent listed.

(a) 10% (b) 2% (c) 25% (d) 0.5%

DECREASING EXPONENTIAL MODELS

If quantity *Q* is known to decrease by a fixed percentage *p*, in decimal form, then *Q* can be modeled by

$$Q(t) = Q_0(1-p)^t$$

where Q_0 represents the amount of *Q* present at $t = 0$ and *t* represents time.

***Exercise* #4:** If the population of a town is decreasing by 4% per year and started with 12,500 residents, which of the following is its projected population in 10 years? Show the exponential model you use to solve this problem.

(1) 9,230 (3) 18,503

(2) 76 (4) 8,310

____

***Exercise* #5:** The stock price of WindpowerInc is increasing at a rate of 4% per week. Its initial value was $20 per share. On the other hand, the stock price in GerbilEnergy is crashing (losing value) at a rate of 11% per week. If its price was $120 per share when Windpower was at $20, after how many weeks will the stock prices be the same? Model both stock prices using exponential functions. Then, find when the stock prices will be equal graphically. Draw a well labeled graph to justify your solution.

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Exponential Modeling with Percent Growth and Decay
Common Core Algebra II Homework

Applications

1. If \$130 is invested in a savings account that earns 4% interest per year, which of the following is closest to the amount in the account at the end of 10 years?

(1) \$218 (3) \$168

(2) \$192 (4) \$324 __________

2. A population of 50 fruit flies is increasing at a rate of 6% per day. Which of the following is closest to the number of days it will take for the fruit fly population to double?

(1) 18 (3) 12

(2) 6 (4) 28 __________

3. If a radioactive substance is quickly decaying at a rate of 13% per hour approximately how much of a 200 pound sample remains after one day?

(1) 7.1 pounds (3) 25.6 pounds

(2) 2.3 pounds (4) 15.6 pounds __________

4. A population of llamas stranded on a dessert island is decreasing due to a food shortage by 6% per year. If the population of llamas started out at 350, how many are left on the island 10 years later?

(1) 257 (3) 102

(2) 58 (4) 189 __________

5. Which of the following equations would model a population with an initial size of 625 that is growing at an annual rate of 8.5%?

(1) $P = 625(8.5)^t$ (3) $P = 1.085^t + 625$

(2) $P = 625(1.085)^t$ (4) $P = 8.5t^2 + 625$ __________

6. The acceleration of an object falling through the air will decrease at a rate of 15% per second due to air resistance. If the initial acceleration due to gravity is 9.8 meters per second per second, which of the following equations best models the acceleration *t* seconds after the object begins falling?

(1) $a = 15 - 9.8t^2$ (3) $a = 9.8(1.15)^t$

(2) $a = \frac{9.8}{15t}$ (4) $a = 9.8(0.85)^t$ __________

7. Red Hook has a population of 6,200 people and is growing at a rate of 8% per year. Rhinebeck has a population of 8,750 and is growing at a rate of 6% per year. In how many years, to the nearest year, will Red Hook have a greater population than Rhinebeck? Show the equation or inequality you are solving and solve it graphically.

8. A warm glass of water, initially at 120 degrees Fahrenheit, is placed in a refrigerator at 34 degrees Fahrenheit and its temperature is seen to decrease according to the exponential function

$$T(h) = 86(0.83)^h + 34$$

(a) Verify that the temperature starts at 120 degrees Fahrenheit by evaluating $T(0)$.

(b) Using your calculator, sketch a graph of T below for all values of h on the interval $0 \le h \le 24$. Be sure to label your y-axis and y-intercept.

(c) After how many hours will the temperature be at 50 degrees Fahrenheit? State your answer to the nearest *hundredth* of an hour. Illustrate your answer on the graph your drew in (b).

REASONING

9. Percents combine in strange ways that don't seem to make sense at first. It would seem that if a population grows by 5% per year for 10 years, then it should grow in total by 50% over a decade. But this isn't true. Start with a population of 100. If it grows at 5% per year for 10 years, what is its population after 10 years? What percent growth does this represent?

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MINDFUL MANIPULATION OF PERCENTS
COMMON CORE ALGEBRA II

Percents and phenomena that grow at a constant percent rate can be challenging, to say the least. This is due to the fact that, unlike linear phenomena, the growth rate indicates a constant multiplier effect instead of a constant additive effect (linear). Because constant percent growth is so common in everyday life (not to mention in science, business, and other fields), it's good to be able to **mindfully manipulate percents**.

***Exercise* #1:** A population of wombats is growing at a constant percent rate. If the population on January 1st is 1027 and a year later is 1079, what is its yearly percent growth rate to the nearest *tenth* of a percent?

***Exercise* #2:** Now let's try to determine what the percent growth in wombat population will be over a decade of time.

(a) After 10 years, what will we have multiplied the original population by, rounded to the nearest hundredth. Show the calculation.

(b) Using your answer from (a), what is the decade percent growth rate?

***Exercise* #3:** Let's stick with our wombats from Exercise #1. Assuming their growth rate is constant over time, what is their monthly growth rate to the nearest tenth of a percent? Assume a constant sized month.

***Exercise* #4:** If a population was growing at a constant rate of 22% every 5 years, then what is its percent growth rate over at 2 year time span? Round to the nearest percent.

(a) First, give an expression that will calculate the single year (or yearly) percent growth rate based on the fact that the population grew 22% in 5 years.

(b) Now use this expression to calculate the percent growth over 2 years.

***Exercise* #5:** World oil reserves (the amount of oil unused in the ground) are depleting at a constant 2% per year. We would like to determine what the percent decline will be over the next 20 years based on this 2% yearly decline.

(a) Write and evaluate an expression for what we would multiply the initial amount of oil by after 20 years.

(b) Use your answer to (a) to determine the percent decline after 20 years. Be careful! Round to the nearest percent.

***Exercise* #6:** A radioactive substance's half-life is the amount of time needed for half (or 50%) of the substance to decay. Let's say we have a radioactive substance with a half-life of 20 years.

(a) What percent of the substance would be radioactive after 40 years?

(b) What percent of the substance would be radioactive after only 10 years? Round to the nearest tenth of a percent.

(c) What percent of the substance would be radioactive after only 5 years? Round to the nearest tenth of a percent.

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Mindful Manipulation of Percents
Common Core Algebra II Homework

Applications

1. A quantity is growing at a constant 3% yearly rate. Which of the following would be its percent growth after 15 years?

(1) 45% (3) 56%

(2) 52% (4) 63%

____

2. If a credit card company charges 13.5% yearly interest, which of the following calculations would be used in the process of calculating the monthly interest rate?

(1) $\frac{0.135}{12}$ (3) $(1.135)^{12}$

(2) $\frac{1.135}{12}$ (4) $(1.135)^{1/12}$

____

3. The county debt is growing at an annual rate of 3.5%. What percent rate is it growing at per 2 years? Per 5 years? Per decade? Show the calculations that lead to each answer. Round each to the nearest tenth of a percent.

4. A population of llamas is growing at a constant yearly rate of 6%. At what rate is the llama population growing per month? Please assume all months are equally sized and that there are 12 of these per year. Round to the nearest tenth of a percent.

5. Shana is trying to increase the number of calories she burns by 5% per day. By what percent is she trying to increase per week? Round to the nearest tenth of a percent.

6. If a bank account doubles in size every 5 years, then by what percent does it grow after only 3 years? Round to the nearest tenth of a percent. Hint: First write an expression that would calculate its growth rate after a single year.

7. An object's speed decreases by 5% for each minute that it is slowing down. Which of the following is closest to the percent that its speed will decrease over half-an hour?

(1) 21%   (3) 48%

(2) 79%   (4) 150%

______

8. Over the last 10 years, the price of corn has decreased by 25% per bushel.

(a) Assuming a steady percent decrease, by what percent does it decrease each year? Round to the nearest tenth of a percent.

(b) Assuming this percent continues, by what percent will the price of corn decrease by after 50 years? Show the calculation that leads to your answer.. Round to the nearest percent.

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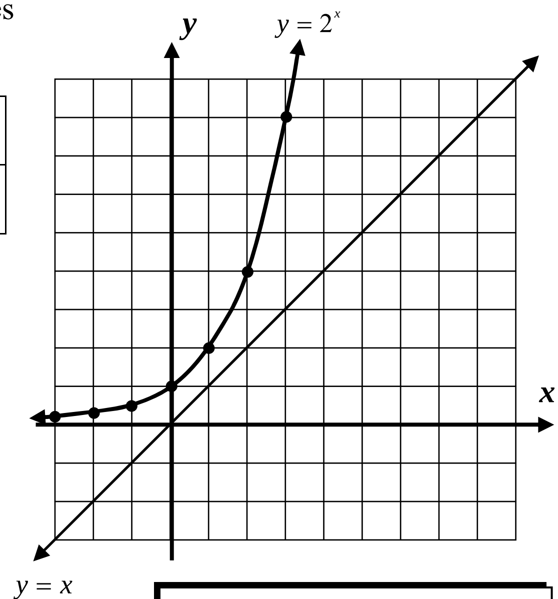
INTRODUCTION TO LOGARITHMS
COMMON CORE ALGEBRA II

Exponential functions are of such importance to mathematics that their inverses, functions that "reverse" their action, are important themselves. These functions, known as **logarithms**, will be introduced in this lesson.

***Exercise* #1:** The function $f(x)=2^x$ is shown graphed on the axes below along with its table of values.



x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

(a) Is this function one-to-one? Explain your answer.

(b) Based on your answer from part (a), what must be true about the inverse of this function?

(c) Create a table of values below for the inverse of $f(x)=2^x$ and plot this graph on the axes given.

Notice that, as always, the graphs of $f(x)$ and $f^{-1}(x)$ are symmetric across $y=x$

x							
$f^{-1}(x)$							

(d) What would be the first step to find an equation for this inverse algebraically? Write this step down and then stop.

<u>Defining Logarithmic Functions</u> – The function $y=\log_b x$ is the name we give the inverse of $y=b^x$. For example, $y=\log_2 x$ is the inverse of $y=2^x$. Based on *Exercise* #1(d), we can write an **equivalent exponential equation** for each logarithm as follows:

$$y=\log_b x \text{ is the same as } b^y=x$$

Based on this, we see that a logarithm gives as its output (y-value) the exponent we must raise b to in order to produce its input (x-value).

***Exercise* #2:** Evaluate the following logarithms. If needed, write an equivalent exponential equation. Do as many as possible without the use of your calculator.

(a) $\log_2 8$ (b) $\log_4 16$ (c) $\log_5 625$ (d) $\log_{10} 100,000$

(e) $\log_6\left(\frac{1}{36}\right)$ (f) $\log_2\left(\frac{1}{16}\right)$ (g) $\log_5 \sqrt{5}$ (h) $\log_3 \sqrt[5]{9}$

It is critically important to understand that logarithms **give exponents as their outputs**. We will be working for multiple lessons on logarithms and a basic understanding of their inputs and outputs is critical.

***Exercise* #3:** If the function $y=\log_2(x+8)+9$ was graphed in the coordinate plane, which of the following would represent its *y*-intercept?

(1) 12 (3) 8

(2) 13 (4) 9

____

***Exercise* #4:** Between which two consecutive integers must $\log_3 40$ lie?

(1) 1 and 2 (3) 3 and 4

(2) 2 and 3 (4) 4 and 5

____

Calculator Use and Logarithms – Most calculators only have two logarithms that they can evaluate directly. One of them, $\log_{10} x$, is so common that it is actually called the **common log** and typically is written without the base 10.

$$\log x = \log_{10} x \quad \text{(The Common Log)}$$

***Exercise* #5:** Evaluate each of the following using your calculator.

(a) $\log 100$ (b) $\log\left(\frac{1}{1000}\right)$ (c) $\log\sqrt{10}$

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Introduction to Logarithms
Common Core Algebra II Homework

Fluency

1. Which of the following is equivalent to $y = \log_7 x$?

(1) $y = x^7$ (3) $x = 7^y$

(2) $x = y^7$ (4) $y = x^{1/7}$

____

2. If the graph of $y = 6^x$ is reflected across the line $y = x$ then the resulting curve has an equation of

(1) $y = -6^x$ (3) $x = \log_6 y$

(2) $y = \log_6 x$ (4) $x = y^6$

____

3. The value of $\log_5 167$ is closest to which of the following? Hint – guess and check the answers.

(1) 2.67 (3) 4.58

(2) 1.98 (4) 3.18

____

4. Which of the following represents the *y*-intercept of the function $y = \log(x + 1000) - 8$?

(1) −8 (3) 3

(2) −5 (4) 5

____

5. Determine the value for each of the following logarithms. (Easy)

(a) $\log_2 32$ (b) $\log_7 49$ (c) $\log_3 6561$ (d) $\log_4 1024$

6. Determine the value for each of the following logarithms. (Medium)

(a) $\log_2\left(1/64\right)$ (b) $\log_3(1)$ (c) $\log_5\left(1/25\right)$ (d) $\log_7\left(1/343\right)$

7. Determine the value for each of the following logarithms. Each of these will have non-integer, fractional answers. (Difficult)

(a) $\log_4 2$ (b) $\log_4 8$ (c) $\log_5 \sqrt[3]{5}$ (d) $\log_2 \sqrt[5]{4}$

8. Between what two consecutive integers must the value of $\log_4 7342$ lie? Justify your answer.

9. Between what two consecutive integers must the value of $\log_5\left(\frac{1}{500}\right)$ lie? Justify your answer.

APPLICATIONS

10. In chemistry, the pH of a solution is defined by the equation $\text{pH} = -\log(H)$ where H represents the concentration of hydrogen ions in the solution. Any solution with a pH less than 7 is considered acidic and any solution with a pH greater than 7 is considered basic. Fill in the table below. Round your pH's to the nearest *tenth* of a unit.

Substance	**Concentration of Hydrogen**	**pH**	**Basic or Acidic?**
Milk	1.6×10^{-7}		
Coffee	1.3×10^{-5}		
Bleach	2.5×10^{-13}		
Lemmon Juice	7.9×10^{-2}		
Rain	1.6×10^{-6}		

REASONING

11. Can the value of $\log_2(-4)$ be found? What about the value of $\log_2 0$? Why or why not? What does this tell you about the domain of $\log_b x$?

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Graphs of Logarithms
Common Core Algebra II

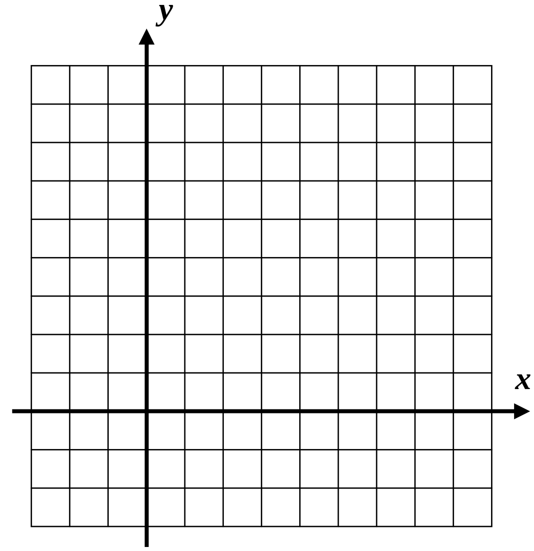
The vast majority of logarithms that are used in the real world have bases greater than one; the pH scale that we saw on the last homework assignment is a good example. In this lesson we will further explore graphs of these logarithms, including their construction, transformations, and domains and ranges.

***Exercise* #1:** Consider the logarithmic function $y = \log_3 x$ and its inverse $y = 3^x$.

(a) Construct a table of values for $y = 3^x$ and then use this to construct a table of values for the function $y = \log_3 x$.

x	-2	-1	0	1	2
$y = 3^x$					

x					
$y = \log_3 x$					



(b) Graph $y = 3^x$ and $y = \log_3 x$ on the grid given. Label with equations.

(c) State the natural domain and range of $y = 3^x$ and $y = \log_3 x$.

$y = 3^x$

Domain:

Range:

$y = \log_3 x$

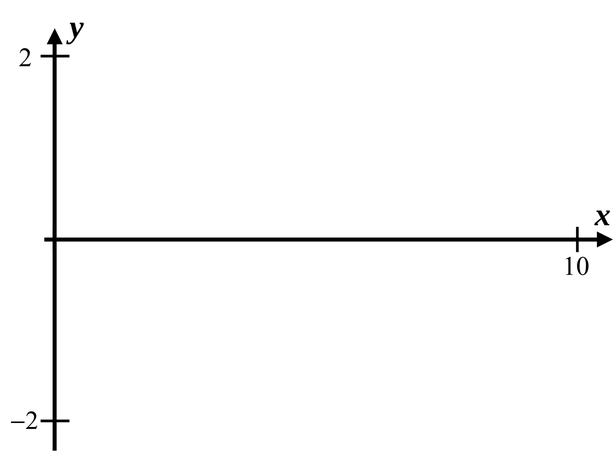
Domain:

Range:

***Exercise* #2:** Using your calculator, sketch the graph of $y = \log_{10} x$ on the axes below. Label the x-intercept. State the domain and range of $y = \log_{10} x$.

Domain:

Range:



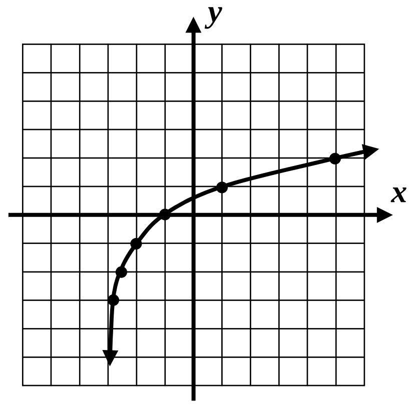
***Exercise* #3:** Which of the following equations describes the graph shown below? Show or explain how you made your choice.

(1) $y=\log_3(x+2)-1$

(2) $y=\log_2(x-3)+1$

(3) $y=\log_2(x+3)-1$

(4) $y=\log_3(x+3)-1$



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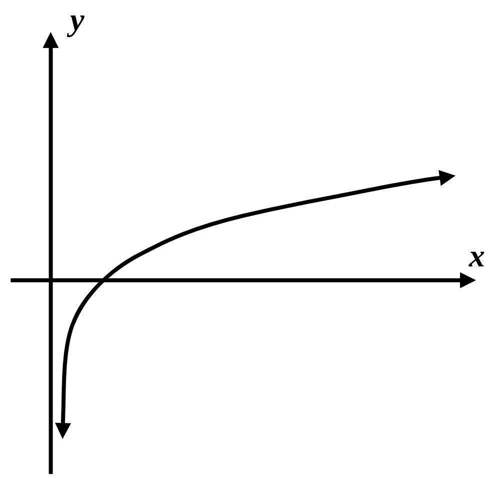
The fact that finding the logarithm of a non-positive number (negative or zero) is not possible in the real number system allows us to find the domains of a variety of logarithmic functions.

***Exercise* #4:** Determine the domain of the function $y=\log_2(3x-4)$. State your answer in set-builder notation.

All logarithms with bases larger than 1 are **always increasing**. This increasing nature can be seen by calculating their **average rate of change**.

***Exercise* #5:** Consider the common log, or log base 10, $f(x)=\log(x)$.

(a) Set up and evaluate an expression for the average rate of change of $f(x)$ over the interval $1\le x\le 10$



(b) Set up and evaluate an expression for the average rate of change of $f(x)$ over the interval $1\le x\le 100$.

(c) What do these two answers tell you about the changing slope of this function?

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GRAPHS OF LOGARITHMS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. The domain of $y = \log_3(x+5)$ in the real numbers is

(1) $\{x \mid x > 0\}$ (3) $\{x \mid x > 5\}$

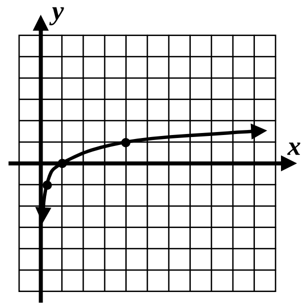
(2) $\{x \mid x > -5\}$ (4) $\{x \mid x \geq -4\}$

____

2. Which of the following equations describes the graph shown below?

(1) $y = \log_5 x$ (3) $y = \log_3 x$

(2) $y = \log_2 x$ (4) $y = \log_4 x$



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3. Which of the following represents the *y*-intercept of the function $y = \log_2(32 - x) - 1$?

(1) 8 (3) -1

(2) -4 (4) 4

____

4. Which of the following values of *x* is *not* in the domain of $f(x) = \log_5(10 - 2x)$?

(1) -3 (3) 5

(2) 0 (4) 4

____

5. Which of the following is true about the function $y = \log_4(x+16) - 1$?

(1) It has an *x*-intercept of 4 and a *y*-intercept of -1.

(2) It has *x*-intercept of -12 and a *y*-intercept of 1.

(3) It has an *x*-intercept of -16 and a *y*-intercept of 1.

(4) It has an *x*-intercept of -16 and a *y*-intercept of -1.

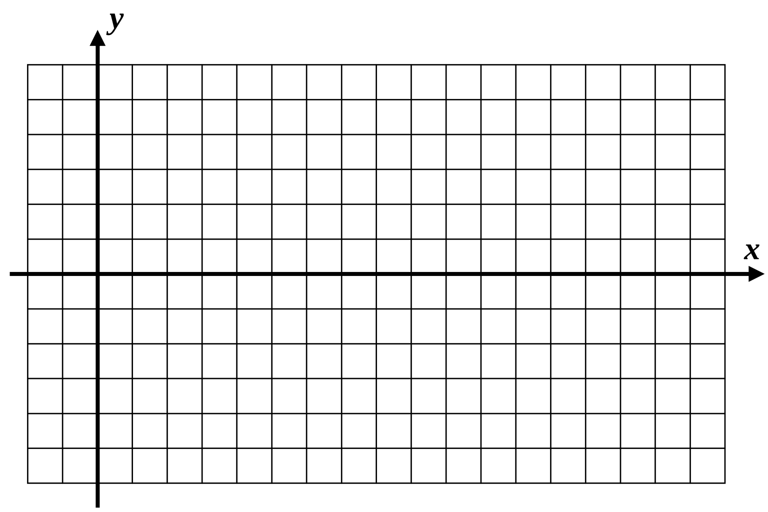
____

6. Determine the domains of each of the following logarithmic functions. State your answers using any accepted notation. Be sure to show the inequality that you are solving to find the domain and the work you use to solve the inequality.

(a) $y = \log_5(2x-1)$

(b) $y = \log(6-x)$

7. Graph the logarithmic function $y = \log_4 x$ on the graph paper given. For a method, see *Exercise* #1.



REASONING

8. Logarithmic functions whose bases are larger than 1 tend to increase **very slowly** as x increases. Let's investigate this for $f(x) = \log_2(x)$.

(a) Find the value of $f(1)$, $f(2)$, $f(4)$, and $f(8)$ without your calculator.

(b) For what value of x will $\log_2(x) = 10$? For what value of x will $\log_2(x) = 20$?

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LOGARITHM LAWS
COMMON CORE ALGEBRA II

Logarithms have properties, just as exponents do, that are important to learn because they allow us to solve a variety of problems where logarithms are involved. Keep in mind that since logarithms give exponents, the laws that govern them should be similar to those that govern exponents. Below is a summary of these laws.

EXPONENT AND LOGARITHM LAWS

LAW	EXPONENT VERSION	LOGARITHM VERSION
Product	$b^x \cdot b^y = b^{x+y}$	$\log_b(x \cdot y) = \log_b x + \log_b y$
Quotient	$\dfrac{b^x}{b^y} = b^{x-y}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
Power	$\left(b^x\right)^y = b^{x \cdot y}$	$\log_b\left(x^y\right) = y \cdot \log_b x$

***Exercise* #1:** Which of the following is equal to $\log_3(9x)$?

(1) $\log_3 2 + \log_3 x$ (3) $2 + \log_3 x$

(2) $2\log_3 x$ (4) $x + \log_3 2$

____

***Exercise* #2:** The expression $\log\left(x^2/1000\right)$ can be written in equivalent form as

(1) $2\log x - 3$ (3) $2\log x - 6$

(2) $\log 2x - 3$ (4) $\log 2x - 6$

____

***Exercise* #3:** If $a = \log 3$ and $b = \log 2$ then which of the following correctly expresses the value of $\log 12$ in terms of a and b?

(1) $a^2 + b$ (3) $2a + b$

(2) $a + b^2$ (4) $a + 2b$

____

***Exercise* #4:** Which of the following is equivalent to $\log_2\left(\sqrt{x}/y^5\right)$?

(1) $\sqrt{\log_2 x} - 5\log_2 y$ (3) $\dfrac{1}{2}\log_2 x - 5\log_2 y$

(2) $2\log_2 x + 5\log_2 y$ (4) $2\log_2 x - 5\log_2 y$

____

***Exercise* #5:** The value of $\log_3\left(\sqrt{5}/27\right)$ is equal to

(1) $\frac{\log_3 5-6}{2}$ (3) $\frac{\log_3 5-3}{2}$

(2) $2\log_3 5+3$ (4) $2\log_3 5-3$ ______

***Exercise* #6:** If $f(x)=\log(x)$ and $g(x)=100x^3$ then $f(g(x))=$

(1) $100\log x$ (3) $300\log x$

(2) $6+\log x$ (4) $2+3\log x$

______

***Exercise* #7:** The logarithmic expression $\log_2\sqrt{32x^7}$ can be rewritten as

(1) $\sqrt{\log_2 35x}$ (3) $\sqrt{5+7\log_2 x}$

(2) $\frac{5+7\log_2 x}{2}$ (4) $\frac{35+\log_2 x}{2}$

______

***Exercise* #8:** If $\log 7=k$ then $\log(4900)$ can be written in terms of k as

(1) $2(k+1)$ (3) $2(k-3)$

(2) $2k-1$ (4) $2k+1$ ______

The logarithm laws are important for future study in mathematics and science. Being fluent with them is essential. Arguably, the most important of the three laws is the power law. In the next exercise, we will examine it more closely.

***Exercise* #9:** Consider the expression $\log_2\left(8^x\right)$.

(a) Using the third logarithm law (the Product Law), rewrite this as equivalent product and simplify.

(b) Test the equivalency of these two expressions for $x=0, 1,$ and 2.

(c) Show that $\log_2\left(8^x\right)=3x$ by rewriting 8 as 2^3.

Name: ____________________________________ Date: __________________

LOGARITHM LAWS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following is not equivalent to $\log 36$?

(1) $\log 2+\log 18$ (3) $\log 30+\log 6$

(2) $2\log 6$ (4) $\log 4+\log 9$ __________

2. The $\log_3 20$ can be written as

(1) $2\log_3 2+\log_3 5$ (3) $\log_3 15+\log_3 5$

(2) $2\log_3 10$ (4) $2\log_3 4+3\log_3 4$ __________

3. Which of the following is equivalent to $\log\left(x^3/\sqrt[3]{y}\right)$?

(1) $\log x-\log y$ (3) $3\log x-\dfrac{1}{3}\log y$

(2) $9\log(x-y)$ (4) $\log(3x)-\log\left(y/3\right)$ __________

4. The difference $\log_2(3)-\log_2(12)$ is equal to

(1) -2 (3) $\dfrac{1}{4}$

(2) $-\dfrac{1}{2}$ (4) 4 __________

5. If $\log 5=p$ and $\log 2=q$ then $\log 200$ can be written in terms of p and q as

(1) $4p+q$ (3) $2(p+q)$

(2) $2p+3q$ (4) $3p+2q$

____

6. When rounded to the nearest hundredth, $\log_3 7 = 1.77$. Which of the following represents the value of $\log_3 63$ to the nearest *hundredth*? Hint: write 63 as a product involving 7.

(1) 3.54 (3) 3.77

(2) 8.77 (4) 15.93

7. The expression $4\log x - \frac{1}{2}\log y + 3\log z$ can be rewritten equivalently as

(1) $\log\left(\frac{x^4z^3}{\sqrt{y}}\right)$ (3) $\log\left(\frac{x^4z^3}{2y}\right)$

(2) $\log\left(\frac{6xz}{y}\right)$ (4) $\log\left(\frac{6x^4z^3}{y}\right)$

8. If $k = \log_2 3$ then $\log_2 48 =$

(1) $2k+3$ (3) $k+8$

(2) $3k+1$ (4) $k+4$

9. If $g(x) = 8x^6$ and $f(x) = \log_4(2x)$ then $f(g(x)) = ?$

(1) $4\log_4 x + 1$ (3) $2(3\log_4 x + 1)$

(2) $3(\log_4 x + 2)$ (4) $6\log_4 x + 4$

REASONING

10. Consider the exponential equation $4^x = 30$.

(a) Between what two consecutive integers must the solution to this equation lie? Explain your reasoning.

(b) Write $\log\left(4^x\right)$ as an equivalent product using the third logarithm law.

(c) The solution to the original equation is $x = \frac{\log(30)}{\log(4)}$, can you see why based on (b)? Evaluate this expression and check to see it is correct.

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SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
COMMON CORE ALGEBRA II

Earlier in this unit, we used the **Method of Common Bases** to solve exponential equations. This technique is quite limited, however, because it requires the two sides of the equation to be expressed using the same base. A more general method utilizes our calculators and the third logarithm law:

THE THIRD LOGARITHM LAW

$$\log_b\left(a^x\right) = x\log_b a$$

***Exercise* #1:** Solve: $4^x = 8$ using (a) common bases and (b) the logarithm law shown above.

(a) Method of Common Bases

(b) Logarithm Approach

The beauty of this logarithm law is that it removes the variable from the exponent. This law, in combination with the logarithm base 10, the **common log**, allows us to solve almost any exponential equation using calculator technology.

***Exercise* #2:** Solve each of the following equations for the value of *x*. Round your answers to the nearest *hundredth*.

(a) $5^x = 18$

(b) $4^x = 100$

(c) $2^x = 1560$

These equations can become more complicated, but each and every time we will use the logarithm law to transform an exponential equation into one that is more familiar (linear only for now)

***Exercise* #3:** Solve each of the following equations for *x*. Round your answers to the nearest *hundredth*.

(a) $6^{x+3} = 50$

(b) $(1.03)^{\frac{x}{2}-5} = 2$

Now that we are familiar with this method, we can revisit some of our exponential models from earlier in the unit. Recall that for an exponential function that is growing:

> If quantity Q is known to increase by a fixed percentage p, in decimal form, then Q can be modeled by
>
> $$Q(t) = Q_0(1+p)^t$$
>
> where Q_0 represents the amount of Q present at $t = 0$ and t represents time.

***Exercise* #4:** A biologist is modeling the population of bats on a tropical island. When he first starts observing them, there are 104 bats. The biologist believes that the bat population is growing at a rate of 3% per year.

(a) Write an equation for the number of bats, $B(t)$, as a function of the number of years, t, since the biologist started observing them.

(b) Using your equation from part (a), algebraically determine the number of years it will take for the bat population to reach 200. Round your answer to the nearest year.

***Exercise* #5:** A stock has been declining in price at a steady pace of 5% per week. If the stock started at a price of \$22.50 per share, determine algebraically the number of weeks it will take for the price to reach \$10.00. Round your answer to the nearest week.

As a final discussion, we return to evaluating logarithms using our calculator. Many modern calculators can find a logarithm of any base. Some still only have the common log (base 10) and another that we will soon see. But, we can still express our answers in terms of logarithms.

***Exercise* #6:** Find the solution to each of the following exponential equations in terms of a logarithm with the same base as the exponential equation.

(a) $4(2)^x - 3 = 17$

(b) $17(5)^{x/3} = 4$

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SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following values, to the nearest *hundredth*, solves: $7^x = 500$?

   (1) 3.19 (3) 2.74

   (2) 3.83 (4) 2.17

   ____

2. The solution to $2^{x/3} = 52$, to the nearest *tenth*, is which of the following?

   (1) 7.3 (3) 11.4

   (2) 9.1 (4) 17.1

   ____

3. To the nearest *hundredth*, the value of *x* that solves $5^{x-4} = 275$ is

   (1) 6.73 (3) 8.17

   (2) 5.74 (4) 7.49

   ____

4. Solve each of the following exponential equations. Round each of your answers to the nearest *hundredth*.

   (a) $9^{x-3} = 250$ (b) $50(2)^x = 1000$ (c) $5^{x/10} = 35$

5. Solve each of the following exponential equations. Be careful with your use of parentheses. Express each answer to the nearest *hundredth*.

   (a) $6^{2x-5} = 300$ (b) $\left(1/2\right)^{\frac{x}{3}+1} = 1/6$ (c) $500(1.02)^{\frac{x}{12}} = 2300$

APPLICATIONS

6. The population of Red Hook is growing at a rate of 3.5% per year. If its current population is 12,500, in how many years will the population exceed 20,000? Round your answer to the nearest year. Only an *algebraic* solution is acceptable.

7. A radioactive substance is decaying such that 2% of its mass is lost every year. Originally there were 50 kilograms of the substance present.

   (a) Write an equation for the amount, A, of the substance left after t-years.

   (b) Find the amount of time that it takes for only half of the initial amount to remain. Round your answer to the nearest tenth of a year.

REASONING

8. If a population doubles every 5 years, how many years will it take for the population to increase by 10 times its original amount?

   First: If the population gets multiplied by 2 every 5 years, what does it get multiplied by each year? Use this to help you answer the question.

9. Find the solution to the general exponential equation $a(b)^{cx} = d$, in terms of the constants a, c, d and the logarithm of base b. Think about reversing the order of operations in order to solve for x.

Name: ____________________________________ Date: __________________

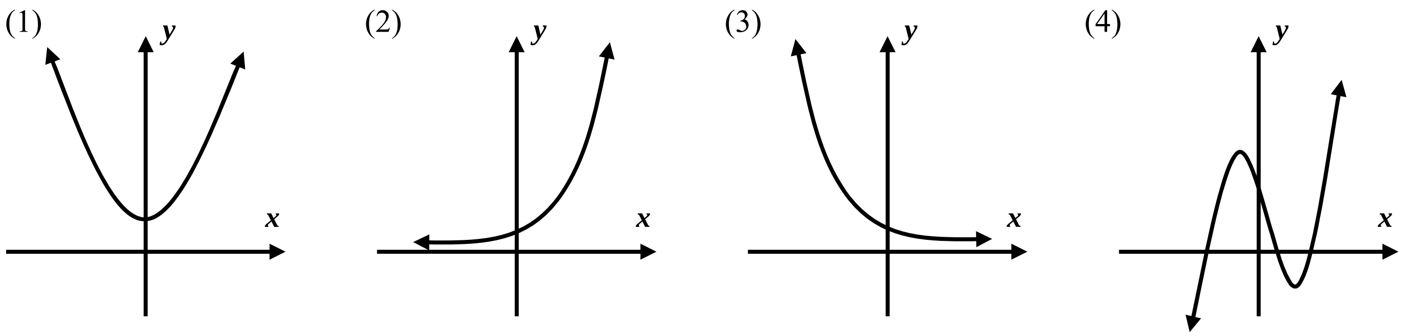
THE NUMBER e AND THE NATURAL LOGARITHM
COMMON CORE ALGEBRA II

There are many numbers in mathematics that are more important than others because they find so many uses in either mathematics or science. Good examples of important numbers are 0, 1, i, and π. In this lesson you will be introduced to an important number given the letter **e** for its "inventor" Leonhard Euler (1707-1783). This number plays a crucial role in Calculus and more generally in modeling exponential phenomena.

THE NUMBER e

1. Like π, e is irrational. 2. $e \approx 2.72$ 3. Used in Exponential Modeling

***Exercise* #1:** Which of the graphs below shows $y = e^x$? Explain your choice. Check on your calculator.



Explanation:

____

Very often e is involved in exponential modeling of both increasing and decreasing quantities. The creation of these models is beyond the scope of this course, but we can still work with them.

***Exercise* #2:** A population of llamas on a tropical island can be modeled by the equation $P = 500e^{0.035t}$, where t represents the number of years since the llamas were first introduced to the island.

(a) How many llamas were initially introduced at $t = 0$. Show the calculation that leads to your answer.

(b) Algebraically determine the number of years for the population to reach 600. Round your answer to the nearest *tenth* of a year.

Because of the importance of $y = e^x$, its **inverse**, known as the **natural logarithm**, is also important.

THE NATURAL LOGARITHM

The inverse of $y = e^x$: $y = \ln x \quad (y = \log_e x)$

The natural logarithm, like all logarithms, gives an exponent as its output. In fact, it gives the power that we must raise **e** to in order to get the input.

***Exercise* #3:** *Without* the use of your calculator, determine the values of each of the following.

(a) $\ln(e)$ (b) $\ln(1)$ (c) $\ln(e^5)$ (d) $\ln\sqrt{e}$

The natural logarithm follows the three basic logarithm laws that all logarithms follow. The following problems give additional practice with these laws.

***Exercise* #4:** Which of the following is equivalent to $\ln\left(\frac{x^3}{e^2}\right)$?

(1) $\ln x + 6$ (3) $3\ln x - 6$

(2) $3\ln x - 2$ (4) $\ln x - 9$

____

***Exercise* #5:** A hot liquid is cooling in a room whose temperature is constant. It's temperature can be modeled using the exponential function shown below. The temperature, *T*, is in degrees Fahrenheit and is a function of the number of minute, *m*, it has been cooling.

$$T(m) = 101e^{-0.03m} + 67$$

(a) What was the initial temperature of the water at $m = 0$. Do without using your calculator.

(b) How do you interpret the statement that $T(60) = 83.7$?

(c) Using the natural logarithm, determine algebraically when the temperature of the liquid will reach $100\ ^\circ\text{F}$. Show the steps in your solution. Round to the nearest tenth of a minute.

(d) On average, how many degrees are lost per minute over the interval $10 \le m \le 30$? Round to the nearest tenth of a degree.

Name: ____________________________________ Date: __________________

THE NUMBER e AND THE NATURAL LOGARITHM
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

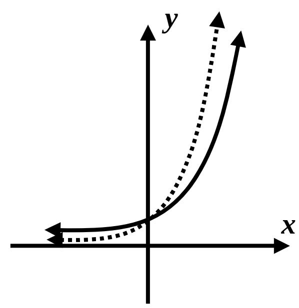
1. Which of the following is closest to the y-intercept of the function whose equation is $y=10e^{x+1}$?

(1) 10 (3) 27

(2) 18 (4) 52

____

2. On the grid below, the solid curve represents $y=e^x$. Which of the following exponential functions could describe the dashed curve? Explain your choice.



(1) $y=\left(\frac{1}{2}\right)^x$ (3) $y=2^x$

(2) $y=e^{-x}$ (4) $y=4^x$

____

3. The logarithmic expression $\ln\left(\frac{\sqrt{e}}{y^3}\right)$ can be rewritten as

(1) $3\ln y-2$ (3) $\frac{\ln y-6}{2}$

(2) $\frac{1-6\ln y}{2}$ (4) $\sqrt{\ln y}-3$

____

4. Which of the following values of t solves the equation $5e^{2t}=15$?

(1) $\frac{\ln 15}{10}$ (3) $2\ln 3$

(2) $\frac{1}{2\ln 5}$ (4) $\frac{\ln 3}{2}$

____

5. At which of the following values of x does $f(x)=2e^{2x}-32$ have a zero?

(1) $\ln\frac{5}{2}$ (3) $\ln 8$

(2) $\ln 4$ (4) $y=\ln\frac{2}{5}$

____

6. For the equation $ae^{ct} = d$, solve for the variable *t* in terms of *a*, *c*, and *d*. Express your answer in terms of the natural logarithm.

APPLICATIONS

7. Flu is spreading exponentially at a school. The number of new flu patients can be modeled using the equation $F = 10e^{0.12d}$, where *d* represents the number of days since 10 students had the flu.

   (a) How many days will it take for the number of new flu patients to equal 50? Determine your answer algebraically using the natural logarithm. Round your answer to the nearest day.

   (b) Find the average rate of change of *F* over the first three weeks, i.e. $0 \le d \le 21$. Show the calculation that leads to your answer. Give proper units and round your answer to the nearest tenth. What is the physical interpretation of your answer?

9. The savings in a bank account can be modeled using $S = 1250e^{.045t}$, where *t* is the number of years the money has been in the account. Determine, to the nearest *tenth* of a year, how long it will take for the amount of savings to double from the initial amount deposited of $1250.

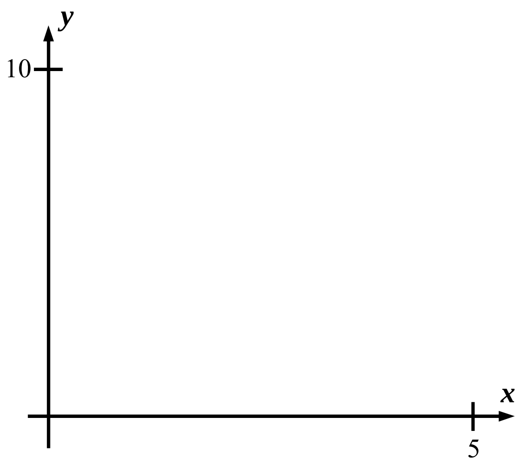
Name: ____________________________________ Date: __________________

NEWTON'S LAW OF COOLING
COMMON CORE ALGEBRA II

The temperature of a cooling liquid in a large room with a steady temperature is a great example of a type of an exponential function. We will explore this today to see how a simple exponential function can be used to build a more complex one.

***Exercise* #1:** Consider the decreasing exponential function $f(x)=8\left(\frac{1}{2}\right)^x$.

(a) Use your calculator to sketch the graph using the window indicated.

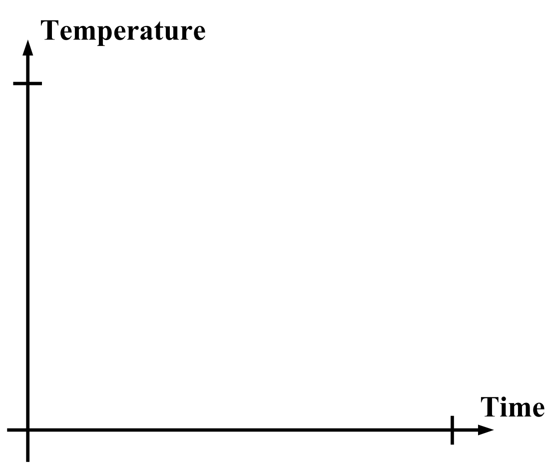


(b) Clearly the value of y gets smaller as x gets larger. Does it ever reach zero? Why or why not?

O.k. So, now let's try to model a liquid's temperature that is cooling in a large room.

***Exercise* #2:** Assume a liquid starts at a temperature of $200\ ^\circ\text{F}$ and begins to cool in a room that is at a steady temperature of $70\ ^\circ\text{F}$.

(a) Draw a rough sketch of what you believe the liquid's temperature function looks like as time increases.



(b) Based on your graph from (a) and your work in Exercise #1, why would an equation of the form $y=a(b)^t$ **not** model this cooling well (assuming that $0<b<1$)? How could we modify this equation to make it model the situation more realistically?

Exercise #3: Let's stick with the same cooling liquid that we had before, i.e. one that starts at $200\ ^\circ\text{F}$ and cools down in a room that is held at a constant temperature of $70\ ^\circ\text{F}$. We will now model this cooling Fahrenheit temperature using the equation $F(t)=a(b)^t+c$, where *a*, *b*, and *c* are all parameters (constants) in the model and *t* is time in minutes.

(a) Which of these constants is equal to 70 and why? Think about the last problem.

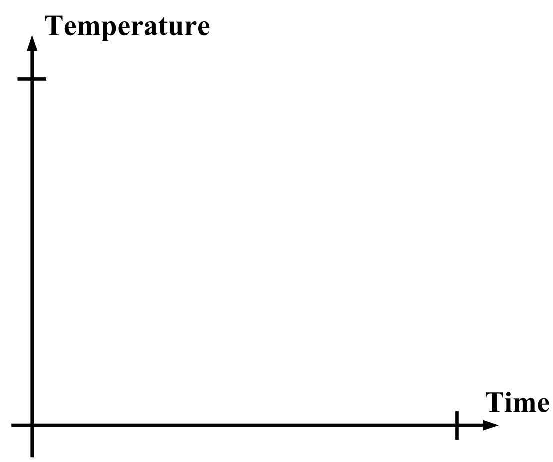
(b) None of these constants is equal to 200, but $T(0)=200$. What constant does this let you solve for? Find its value.

(c) In order to find the value of *b* what additional information would we need?

(d) Determine the value of *b* if the temperature, after 5 minutes, is $153\ ^\circ\text{F}$. Round to the nearest *hundredth*.

(e) What is the temperature of the liquid after half an hour? An hour? Two hours?

(f) Using your calculator, sketch the graph of the liquid's temperature. Decide on an appropriate window and label it on your axes.



(g) Algebraically determine, to the nearest tenth of a minute, when the temperature reaches $100\ ^\circ\text{F}$.

Name: ____________________________________ Date: __________________

NEWTON'S LAW OF COOLING
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. For the function $f(x)=a(b)^x$, where $0<b<1$, what value does the function's output approach as x gets very large?

2. For the function $g(x)=a(b)^x+k$, where $0<b<1$, what value does the function's output approach as x gets very large?

3. Given $f(t)=a(b)^t+c$, which of the following represents the y-intercept of this function? Show how you arrived at your choice.

(1) a (3) c

(2) $a+c$ (4) $b+c$ __________

APPLICATIONS

4. A liquid starts at a temperature of $175\ ^\circ\text{C}$ and cools down in a room held at a constant temperature of $16\ ^\circ\text{C}$. It's temperature can be modeled, as a function of time cooling, by the equation $y=a(b)^x+c$. Which of the following statements is true?

(1) $a=159$ and $c=16$ (3) $a=175$ and $c=16$

(2) $a=16$ and $c=159$ (4) $a=16$ and $c=175$ __________

5. A cooling liquid has a temperature given by the function $T(m)=132(.83)^m+40$, where m is the number of minutes it has been cooling. Which of the following temperatures did the liquid start at?

(1) 40 (3) 172

(2) 92 (4) 132 __________

6. A liquid starts at a temperature of $190\ ^\circ\text{F}$ and cools down in a room held at a constant $65\ ^\circ\text{F}$. After 10 minutes of cooling, it is at a temperature of $92\ ^\circ\text{F}$. The Fahrenheit temperature, *F*, can be modeled as a function of time in minutes, *t*, by the equation:

$$F(t)=a(b)^t+c$$

(a) Determine the values of the parameters *a*, *b*, and *c*. Round the value of *b* to the nearest *hundredth*. State the equation of your final model. Show the work that leads to each of your answers.

(b) Algebraically, determine the number of minutes it will take for the temperature to reach $70\ ^\circ\text{F}$. Round to the nearest *tenth* of a minute.

Reasoning

7. When we model the temperature of a cooling liquid using the equation $T=a(b)^x+c$, we have learned that the value of *c* represents the steady temperature of the room. The quantity $a(b)^x$ does model something physically. Can you determine what it is?

8. **A Warming Liquid** - A liquid is taken out of a refrigerator and placed in a warmer room, where its temperature, in F, increases over time. It can be modeled using the equation $T(m)=74-39(0.87)^m$.

(a) What temperature did the liquid start at? Show the work that leads to your answer.

(b) What is the temperature of the room?