Name: ___________________________________ Date: __________________

SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied **sequences**, which are ordered lists of numbers. Sequences are extremely important in mathematics, both theoretical and applied. A **sequence** is formally defined as **a function that has as its domain the set the set of positive integers**, i.e. $\{1, 2, 3, ..., n\}$.

***Exercise* #1:** A sequence is defined by the equation $a(n) = 2n - 1$.

(a) Find the first three terms of this sequence, denoted by a_1, a_2, and a_3.

(b) Which term has a value of 53?

(c) Explain why there will not be a term that has a value of 70.

Recall that sequences can also be described by using **recursive definitions**. When a sequence is defined recursively, terms are found by operations on previous terms.

***Exercise* #2:** A sequence is defined by the recursive formula: $f(n) = f(n-1) + 5$ with $f(1) = -2$.

(a) Generate the first five terms of this sequence. Label each term with proper subscript notation.

(b) Determine the value of $f(20)$. Hint – think about how many times you have added 5 to -2.

***Exercise* #3:** Determine a recursive definition, in terms of $f(n)$, for the sequence shown below. Be sure to include a starting value.

5, 10, 20, 40, 80, 160, …

***Exercise* #4:** For the recursively defined sequence $t_n = (t_{n-1})^2 + 2$ and $t_1 = 2$, the value of t_4 is

(1) 18 (3) 456

(2) 38 (4) 1446

____

***Exercise* #5:** One of the most well-known sequences is the Fibonacci, which is defined recursively using two previous terms. Its definition is given below.

$$f(n) = f(n-1) + f(n-2) \text{ and } f(1) = 1 \text{ and } f(2) = 1$$

Generate values for $f(3)$, $f(4)$, $f(5)$, and $f(6)$ (in other words, then next four terms of this sequence).

It is often possible to find algebraic formulas for simple series, and this skill should be practiced.

***Exercise* #6:** Find an algebraic formula $a(n)$, similar to that in *Exercise* #1, for each of the following sequences. Recall that the domain that you map from will be the set $\{1, 2, 3, ..., n\}$.

(a) $4, 5, 6, 7, ...$

(b) $2, 4, 8, 16, ...$

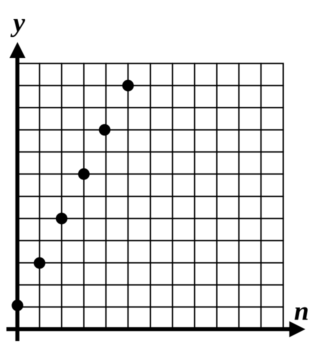
(c) $5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, ...$

(d) $-1, 1, -1, 1, ...$

(e) $10, 15, 20, 25, \ldots$

(f) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...$

***Exercise* #7:** Which of the following would represent the graph of the sequence $a_n = 2n+1$? Explain your choice.

(1)


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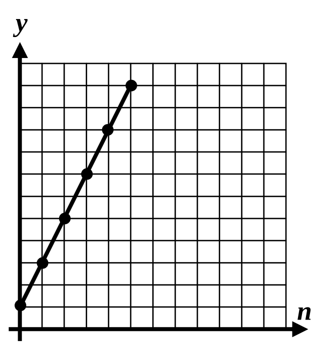

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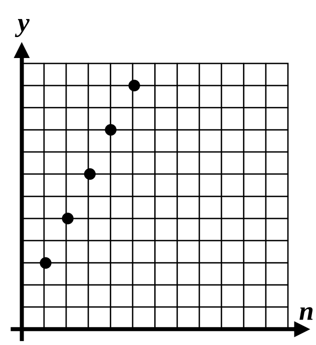

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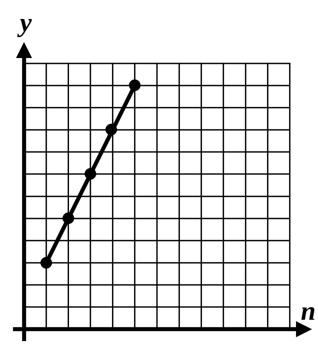

Explanation:

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SEQUENCES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Given each of the following sequences defined by formulas, determine and label the first three terms. A variety of different notations is used below for practice purposes.

(a) $f(n)=7n+2$ (b) $a_n=n^2-5$ (c) $t(n)=\left(\frac{2}{3}\right)^n$ (d) $t_n=\frac{1}{n+1}$

2. Sequences below are defined recursively. Determine and label the **next** three terms of the sequence.

(a) $f(1)=4$ and $f(n)=f(n-1)+8$ (b) $a(n)=a(n-1)\cdot\frac{1}{2}$ and $a(1)=24$

(c) $b_n=b_{n-1}+2n$ with $b_1=5$ (d) $f(n)=2f(n-1)-n^2$ and $f(1)=4$

3. Given the sequence 7, 11, 15, 19, ..., which of the following represents a formula that will generate it?

(1) $a(n)=4n+7$ (3) $a(n)=3n+7$

(2) $a(n)=3n+4$ (4) $a(n)=4n+3$

____

4. A recursive sequence is defined by $a_{n+1}=2a_n-a_{n-1}$ with $a_1=0$ and $a_2=1$. Which of the following represents the value of a_5?

(1) 8 (3) 3

(2) −7 (4) 4

____

5. Which of the following formulas would represent the sequence 10, 20, 40, 80, 160, …

(1) $a_n=10^n$ (3) $a_n=5(2)^n$

(2) $a_n=10(2)^n$ (4) $a_n=2n+10$

____

6. For each of the following sequences, determine an algebraic formula, similar to *Exercise* #4, that defines the sequence.

(a) 5, 10, 15, 20, …

(b) 3, 9, 27, 81, …

(c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

7. For each of the following sequences, state a recursive definition. Be sure to include a starting value or values.

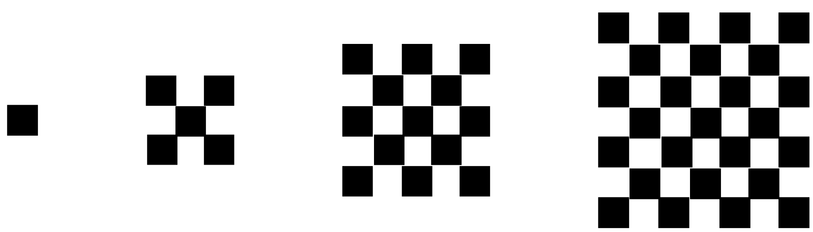
(a) 8, 6, 4, 2, …

(b) 2, 6, 18, 54, …

(c) $2, -2, 2, -2, \ldots$

APPLICATIONS

8. A tiling pattern is created from a single square and then expanded as shown. If the number of squares in each pattern defines a sequences, then determine the number of squares in the seventh pattern. Explain how you arrived at your choice. Can you write a recursive definition for the pattern?



REASONING

9. Consider a sequence defined similarly to the Fibonacci, but with a slight twist:

$$f(n) = f(n-1) - f(n-2) \text{ with } f(1) = 2 \text{ and } f(2) = 5$$

Generate terms $f(3)$, $f(4)$, $f(5)$, $f(6)$, $f(7)$, and $f(8)$. Then, determine the value of $f(25)$.

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ARITHMETIC AND GEOMETRIC SEQUENCES
COMMON CORE ALGEBRA II

In Common Core Algebra I, you studied two particular sequences known as **arithmetic** (based on constant addition to get the next term) and **geometric** (based on constant multiplying to get the next term). In this lesson, we will review the basics of these two sequences.

ARITHMETIC SEQUENCE RECURSIVE DEFINITION

Given $f(1)$, then $f(n)=f(n-1)+d$ or given a_1 then $a_n=a_{n-1}+d$

where d is called the **common difference** and can be positive or negative.

***Exercise* #1:** Generate the next three terms of the given arithmetic sequences.

(a) $f(n)=f(n-1)+6$ with $f(1)=2$

(b) $a_n=a_{n-1}+\frac{1}{2}$ and $a_1=\frac{3}{2}$

***Exercise* #2:** For some number *t*, the first three terms of an arithmetic sequence are $2t$, $5t-1$, and $6t+2$. What is the numerical value of the fourth term? Hint: first set up an equation that will solve for *t*.

It is important to be able to determine a general term of an arithmetic sequence based on the value of the index variable (the subscript). The next exercise walks you through the thinking process involved.

***Exercise* #3:** Consider $a_n=a_{n-1}+3$ with $a_1=5$.

(a) Determine the value of a_2, a_3, and a_4.

(b) How many times was 3 added to 5 in order to produce a_4?

(c) Use your result from part (b) to quickly find the value of a_{50}.

(d) Write a formula for the n^{th} term of an arithmetic sequence, a_n, based on the first term, a_1, d and n.

***Exercise* #4:** Given that $a_1 = 6$ and $a_4 = 18$ are members of an arithmetic sequence, determine the value of a_{20}.

Geometric sequences are defined very similarly to arithmetic, but with a multiplicative constant instead of an additive one.

GEOMETRIC SEQUENCE RECURSIVE DEFINITION

Given $f(1)$ then $f(n) = f(n-1)\cdot r$ or given a_1, then $a_n = a_{n-1}\cdot r$

where r is called the **common ratio** and can be positive or negative and is often fractional.

***Exercise* #5:** Generate the next three terms of the geometric sequences given below.

(a) $a_1 = 4$ and $r = 2$

(b) $f(n) = f(n-1)\cdot \frac{1}{3}$ with $f(1) = 9$

(c) $t_n = t_{n-1}\cdot\sqrt{2}$ with $t_1 = 3\sqrt{2}$

And, like arithmetic, we also need to be able to determine any given term of a geometric sequence based on the first value, the common ratio, and the index.

***Exercise* #6:** Consider $a_1 = 2$ and $a_n = a_{n-1}\cdot 3$.

(a) Generate the value of a_4.

(b) How many times did you need to multiply 2 by 3 in order to find a_4.

(c) Determine the value of a_{10}.

(d) Write a formula for the n^{th} term of a geometric sequence, a_n, based on the first term, a_1, r and n.

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ARITHMETIC AND GEOMETRIC SEQUENCES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Generate the next **three** terms of each **arithmetic sequence** shown below.

   (a) $a_1 = -2$ and $d = 4$ (b) $f(n) = f(n-1) - 8$ with $f(1) = 10$ (c) $a_1 = 3,\ a_2 = 1$

2. In an arithmetic sequence $t_n = t_{n-1} + 7$. If $t_1 = -5$ determine the values of t_4 and t_{20}. Show the calculations that lead to your answers.

3. If $x+4$, $2x+5$, and $4x+3$ represent the first three terms of an arithmetic sequence, then find the value of x. What is the fourth term?

4. If $f(1) = 12$ and $f(n) = f(n-1) - 4$ then which of the following represents the value of $f(40)$?

   (1) -148 (3) -144

   (2) -140 (4) -172 ________

5. In an arithmetic sequence of numbers $a_1 = -4$ and $a_6 = 46$. Which of the following is the value of a_{12}?

   (1) 120 (3) 92

   (2) 146 (4) 106 ________

6. The first term of an arithmetic sequence whose common difference is 7 and whose 22nd term is given by $a_{22} = 143$ is which of the following?

   (1) -25 (3) 7

   (2) -4 (4) 28 ________

7. Generate the next **three** terms of each geometric sequence defined below.

(a) $a_1 = -8$ with $r = -1$

(b) $a_n = a_{n-1} \cdot \frac{3}{2}$ and $a_1 = 16$

(c) $f(n) = f(n-1) \cdot -2$ and $f(1) = 5$

8. Given that $a_1 = 5$ and $a_2 = 15$ are the first two terms of a geometric sequence, determine the values of a_3 and a_{10}. Show the calculations that lead to your answers.

9. In a geometric sequence, it is known that $a_1 = -1$ and $a_4 = 64$. The value of a_{10} is

(1) −65,536

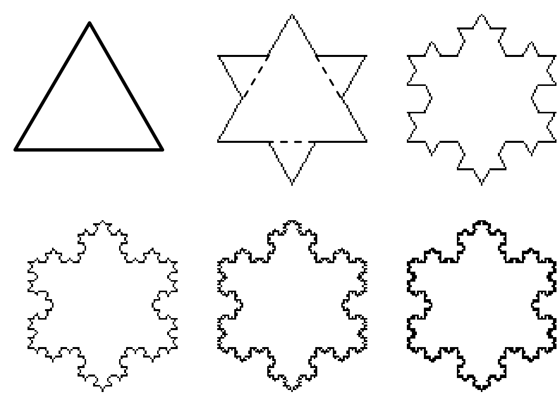
(2) 262,144

(3) 512

(4) −4096

APPLICATIONS

10. The Koch Snowflake is a mathematical shape known as a **fractal** that has many fascinating properties. It is created by repeatedly forming equilateral triangles off of the sides of other equilateral triangles. Its first six iterations are shown to the right. The perimeters of each of the figures form a geometric sequence.



(a) If the perimeter of the first snowflake (the equilateral triangle) is 3, what is the perimeter of the second snow flake? Note: the dashed lines in the second snowflake are not to be counted towards the perimeter. They are only there to show how the snowflake was constructed.

(b) Given that the perimeters form a geometric sequence, what is the perimeter of the sixth snowflake? Express your answer to the nearest tenth.

(c) If the this process was allowed to continue forever, explain why the perimeter would become infinitely large.

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SUMMATION NOTATION
COMMON CORE ALGEBRA II

Much of our work in this unit will concern **adding the terms of a sequence**. In order to specify this addition or summarize it, we introduce a new notation, known as **summation or sigma notation** that will represent these sums. This notation will also be used later in the course when we want to write formulas used in statistics.

SUMMATION (SIGMA) NOTATION

$$\sum_{i=a}^{n} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(n)$$

where ***i*** is called the **index variable**, which starts at a value of *a*, ends at a value of *n*, and moves by unit increments (increase by 1 each time).

***Exercise* #1:** Evaluate each of the following sums.

(a) $\sum_{i=3}^{5} 2i$

(b) $\sum_{k=-1}^{3} k^2$

(c) $\sum_{j=-2}^{2} 2^j$

(d) $\sum_{i=1}^{5} (-1)^i$

(e) $\sum_{k=0}^{2} (2k+1)$

(f) $\sum_{i=1}^{3} i(i+1)$

***Exercise* #2:** Which of represents the value of $\sum_{i=1}^{4} \frac{1}{i}$?

(1) $1/10$

(3) $\frac{25}{12}$

(2) $9/4$

(4) $\frac{31}{24}$

____

***Exercise* #3:** Consider the sequence defined recursively by $a_n = a_{n-1} + 2a_{n-2}$ and $a_1 = 0$ and $a_2 = 1$. Find the value of $\sum_{i=4}^{7} a_i$

It is also good to be able to place sums into sigma notation. These answers, though, will not be unique.

***Exercise* #4:** Express each sum using sigma notation. Use *i* as your index variable. First, consider any patterns you notice amongst the terms involved in the sum. Then, work to put these patterns into a formula and sum.

(a) $9+16+25+\cdots+100$

(b) $-6+-3+0+3+\cdots 15$

(c) $\frac{1}{25}+\frac{1}{5}+1+5+\cdots 625$

***Exercise* #5:** Which of the following represents the sum $3+6+12+24+48$?

(1) $\sum_{i=1}^{5} 3^i$

(2) $\sum_{i=0}^{4} 3(2)^i$

(3) $\sum_{i=0}^{4} 6^{i-1}$

(4) $\sum_{i=3}^{48} i$

____

***Exercise* #6:** Some sums are more interesting than others. Determine the value of $\sum_{i=1}^{99}\left(\frac{1}{i}-\frac{1}{i+1}\right)$. Show your reasoning. This is known as a **telescoping series (or sum)**.

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SUMMATION NOTATION
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Evaluate each of the following. Place any non-integer answer in simplest rational form.

(a) $\sum_{i=2}^{5} 4i$ (b) $\sum_{k=0}^{3} \left(k^2+1\right)$ (c) $\sum_{j=-2}^{0} \left(2j+1\right)$

(d) $\sum_{i=-1}^{3} 2^i$ (e) $\sum_{k=0}^{2} \left(-1\right)^{2k+1}$ (f) $\sum_{i=1}^{3} \log\left(10^i\right)$

(g) $\sum_{n=1}^{4} \frac{n}{n+1}$ (h) $\dfrac{\sum_{i=2}^{4} \left(i+1\right)^2}{\sum_{i=2}^{4} \left(i^2+1\right)}$ (i) $\sum_{k=0}^{3} 256^{\frac{1}{2^k}}$

2. Which of the following is the value of $\sum_{k=0}^{4} \left(4k+1\right)$?

(1) 53 (3) 37

(2) 45 (4) 80

____

3. The sum $\sum_{i=4}^{7} 2^{i-7}$ is equal to

(1) $15/8$ (3) $3/4$

(2) $3/2$ (4) $7/8$

____

4. Write each of the following sums using sigma notation. Use *k* as your index variable. Note, there are many correct ways to write each sum (and even more incorrect ways).

(a) $-2+4+-8+\cdots+64+-128$ (b) $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{81}+\frac{1}{100}$ (c) $4+9+14+\cdots+44+49$

5. Which of the following represents the sum $2+5+10+\cdots+82+101$?

(1) $\sum_{j=1}^{6}(4j-3)$ (3) $\sum_{j=1}^{10}(j^2+1)$

(2) $\sum_{j=3}^{103}(j-2)$ (4) $\sum_{j=0}^{11}(4^j+1)$

______

6. A sequence is defined recursively by the formula $b_n = 4b_{n-1} - 2b_{n-2}$ with $b_1 = 1$ and $b_2 = 3$. What is the value of $\sum_{i=3}^{5} b_i$? Show the work that leads to your answer.

Reasoning

6. A curious pattern occurs when we look at the behavior of the sum $\sum_{k=1}^{n}(2k-1)$.

(a) Find the value of this sum for a variety of values of *n* below:

$n=2$: $\sum_{k=1}^{2}(2k-1)=$ $n=4$: $\sum_{k=1}^{4}(2k-1)=$

$n=3$: $\sum_{k=1}^{3}(2k-1)=$ $n=5$: $\sum_{k=1}^{5}(2k-1)=$

(b) What types of numbers are you summing? What types of numbers are the sums?

(c) Find the value of *n* such that $\sum_{k=1}^{n}(2k-1)=196$.

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ARITHMETIC SERIES
COMMON CORE ALGEBRA II

A **series** is simply the **sum of the terms of a sequence**. The fundamental definition/notion of a series is below.

THE DEFINITION OF A SERIES

If the set $\{a_1, a_2, a_3, ...\}$ represent the elements of a sequence then the series, S_n, is defined by:

$$S_n = \sum_{i=1}^{n} a_i$$

In truth, you have already worked extensively with series in previous lessons almost anytime you evaluated a summation problem.

Exercise #1: Given the arithmetic sequence defined by $a_1 = -2$ and $a_n = a_{n-1} + 5$, then which of the following is the value of $S_5 = \sum_{i=1}^{5} a_i$?

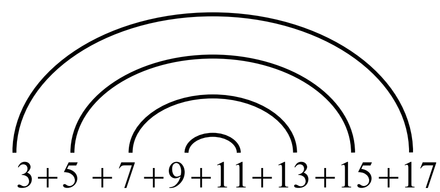
(1) 32 (3) 25

(2) 40 (4) 27 ________

The sums associated with arithmetic sequences, known as **arithmetic series**, have interesting properties, many applications and values that can be predicted with what is commonly known as **rainbow addition**.

Exercise #2: Consider the arithmetic sequence defined by $a_1 = 3$ and $a_n = a_{n-1} + 2$. The series, based on the first eight terms of this sequence, is shown below. Terms have been paired off as shown.

(a) What does each of the paired off sums equal?



(b) Why does it make sense that this sum is constant?

(c) How many of these pairs are there?

(d) Using your answers to (a) and (c) find the value of the sum using a multiplicative process.

(e) Generalize this now and create a formula for an arithmetic series sum based only on its first term, a_1, its last term, a_n, and the number of terms, *n*.

SUM OF AN ARITHMETIC SERIES

Given an arithmetic series with n terms, $\{a_1, a_2, ..., a_n\}$, then its sum is given by:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

***Exercise* #3:** Which of the following is the sum of the first 100 natural numbers? Show the process that leads to your choice.

(1) 5,000 (3) 10,000

(2) 5,100 (4) 5,050 ________

***Exercise* #4:** Find the sum of each arithmetic series described or shown below.

(a) The sum of the sixteen terms given by:

$$-10 + -6 + -2 + \cdots + 46 + 50.$$

(b) The first term is -8, the common difference, d, is 6 and there are 20 terms

(c) The last term is $a_{12} = -29$ and the common difference, d, is -3.

(d) The sum $5 + 8 + 11 + \cdots + 77$.

***Exercise* #5:** Kirk has set up a college savings account for his son, Maxwell. If Kirk deposits \$100 per month in an account, increasing the amount he deposits by \$10 per month each month, then how much will be in the account after 10 years?

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ARITHMETIC SERIES
COMMON CORE ALGEBRA II HOMEWORK

FLUENCY

1. Which of the following represents the sum of $3+10+\cdots+87+94$ if the arithmetic series has 14 terms?

   (1) 1,358 (3) 679

   (2) 658 (4) 1,276

   ____

2. The sum of the first 50 natural numbers is

   (1) 1,275 (3) 1,250

   (2) 1,875 (4) 950

   ____

3. If the first and last terms of an arithmetic series are 5 and 27, respectively, and the series has a sum 192, then the number of terms in the series is

   (1) 18 (3) 14

   (2) 11 (4) 12

   ____

4. Find the sum of each arithmetic series described or shown below.

   (a) The sum of the first 100 even, natural numbers.

   (b) The sum of multiples of five from 10 to 75, inclusive.

   (c) A series whose first two terms are -12 and -8, respectively, and whose last term is 124.

   (d) A series of 20 terms whose last term is equal to 97 and whose common difference is five.

5. For an arithmetic series that sums to 1,485, it is known that the first term equals 6 and the last term equals 93. *Algebraically* determine the number of terms summed in this series.

APPLICATIONS

6. Arlington High School recently installed a new black-box theatre for local productions. They only had room for 14 rows of seats, where the number of seats in each row constitutes an arithmetic sequence starting with eight seats and increasing by two seats per row thereafter. How many seats are in the new black-box theatre? Show the calculations that lead to your answer.

7. Simeon starts a retirement account where he will place \$50 into the account on the first month and increasing his deposit by \$5 per month each month after. If he saves this way for the next 20 years, how much will the account contain in principal?

8. The distance an object falls per second while only under the influence of gravity forms an arithmetic sequence with it falling 16 feet in the first second, 48 feet in the second, 80 feet in the third, etcetera. What is the total distance an object will fall in 10 seconds? Show the work that leads to your answer.

9. A large grandfather clock strikes its bell once at 1:00, twice at 2:00, three times at 3:00, etcetera. What is the total number of times the bell will be struck in a day? Use an arithmetic series to help solve the problem and show how you arrived at your answer.

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GEOMETRIC SERIES
COMMON CORE ALGEBRA II

Just as we can sum the terms of an arithmetic sequence to generate an arithmetic series, we can also sum the terms of a geometric sequence tNo generate a **geometric series**.

***Exercise* #1:** Given a geometric series defined by the recursive formula $a_1 = 3$ and $a_n = a_{n-1} \cdot 2$, which of the following is the value of $S_5 = \sum_{i=1}^{5} a_i$?

(1) 106 (3) 93

(2) 75 (4) 35 ________

The sum of a finite number of geometric sequence terms is less obvious than that for an arithmetic series, but can be found nonetheless. The next exercise derives the formula for finding this sum.

***Exercise* #2:** Recall that for a geometric sequence, the *n*th term is given by $a_n = a_1 \cdot r^{n-1}$. Thus, the general form of an geometric series is given below.

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-2} + a_1 r^{n-1}$$

(a) Write an expression below for the product of *r* and S_n.

$r \cdot S_n =$

(b) Find, in simplest form, the value of $S_n - r \cdot S_n$ in terms of $a_1, r,$ and n.

$S_n - r \cdot S_n =$

(c) Write both sides of the equation in (b) in their factored form.

(d) From the equation in part (c), find a formula for S_n in terms of $a_1, r,$ and n.

***Exercise* #3:** Which of the following represents the sum of a geometric series with 8 terms whose first term is 3 and whose common ratio is 4?

(1) 32,756 (3) 42,560

(2) 28,765 (4) 65,535 ________

SUM OF A FINITE GEOMETRIC SERIES

For a geometric series defined by its first term, a_1, and its common ratio, r, the sum of n terms is given by:

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r} \quad \text{or} \quad S_n = \frac{a_1 - a_1 r^n}{1-r}$$

***Exercise* #4:** Find the value of the geometric series shown below. Show the calculations that lead to your final answer.

$$6+12+24+\cdots+768$$

***Exercise* #5:** Maria places $500 at the beginning of each year into an account that earns 5% interest compounded annually. Maria would like to determine how much money is in her account after she has made her $500 deposit at the end of 10 years.

(a) Determine a formula for the amount, $A(t)$, that a given $500 has grown to t-years after it was placed into this account.

(b) At the end of 10 years, which will be worth more: the $500 invested in the first year or the fourth year? Explain by showing how much each is worth at the beginning of the 11th year.

(c) Based on (b), write a geometric sum representing the amount of money in Maria's account after 10 years.

(d) Evaluate the sum in (c) using the formula above.

***Exercise* #6:** A person places 1 penny in a piggy bank on the first day of the month, 2 pennies on the second day, 4 pennies on the third, and so on. Will this person be a millionaire at the end of a 31 day month? Show the calculations that lead to your answer.

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GEOMETRIC SERIES
COMMON CORE ALGEBRA I HOMEWORK

FLUENCY

1. Find the sums of geometric series with the following properties:

(a) $a_1 = 6, r = 3$ and $n = 8$ (b) $a_1 = 20, r = \frac{1}{2}$, and $n = 6$ (c) $a_1 = -5, r = -2,$ and $n = 10$

2. If the geometric series $54 + 36 + \cdots + \frac{128}{27}$ has seven terms in its sum then the value of the sum is

(1) $\frac{4118}{27}$ (3) $\frac{1370}{9}$

(2) $\frac{1274}{3}$ (4) $\frac{8241}{54}$

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3. A geometric series has a first term of 32 and a final term of $-\frac{1}{4}$ and a common ratio of $-\frac{1}{2}$. The value of this series is

(1) 19.75 (3) 22.5

(2) 16.25 (4) 21.25

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4. Which of the following represents the value of $\sum_{i=0}^{8} 256\left(\frac{3}{2}\right)^i$? Think carefully about how many terms this series has in it.

(1) 19,171 (3) 22,341

(2) 12,610 (4) 8,956

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5. A geometric series whose first term is 3 and whose common ratio is 4 sums to 4095. The number of terms in this sum is

(1) 8 (3) 6

(2) 5 (4) 4

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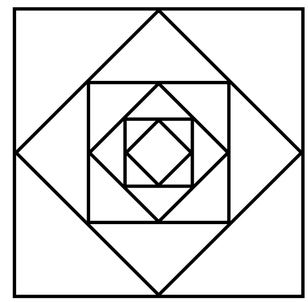
6. Find the sum of the geometric series shown below. Show the work that leads to your answer.

$$27+9+3+\cdots\frac{1}{729}$$

APPLICATIONS

7. In the picture shown at the right, the outer most square has an area of 16 square inches. All other squares are constructed by connecting the midpoints of the sides of the square it is inscribed within. Find the sum of the areas of all of the squares shown. First, consider the how the area of each square relates to the larger square that surrounds (circumscribes) it.



8. A college savings account is constructed so that \$1000 is placed the account on January 1st of each year with a guaranteed 3% yearly return in interest, applied at the end of each year to the balance in the account. If this is repeatedly done, how much money is in the account after the \$1000 is deposited at the beginning of the 19th year? Show the sum that leads to your answer as well as relevant calculations.

9. A ball is dropped from 16 feet above a hard surface. After each time it hits the surface, it rebounds to a height that is $\frac{3}{4}$ of its previous maximum height. What is the total vertical distance, to the nearest foot, the ball has traveled when it strikes the ground for the 10th time? Write out the first five terms of this sum to help visualize.

Name: ___________________________________ Date: _________________

Mortgage Payments
Common Core Algebra II

Mortgages, not just on houses, are large amounts of money borrowed from a bank on which interest is calculated (added on) on a regular (typically month) basis. Regular payments are also made on the amount of money owed so that over time the principal (original amount borrowed) is paid off as well as any interest on the amount owed. This is a complex process that ultimately involves **geometric series**. First, some basics.

***Exercise* #1:** Let's say a person takes out a mortgage for \$200,000 and wants to make payments of \$1500 each month of pay it off. The bank is going to charge this person 4% nominal yearly interest, applied monthly.

(a) What is the amount owed at the end of the first month? Show the calculations that lead to your answer.

(b) How much of the first month's payment went to paying off the principal? How much of it went to paying interest on the loan? Show your calculations.

Amount Towards Principal Amount Towards Interest

(c) Determine the amount owed at the end of the second month. Again, show the calculations that lead to your answer.

(d) The amount owed at the end of a month actually forms a sequence that can be defined recursively. If $a_1 = 200,000$, then define a recursive rule that gives this sequence.

***Exercise* #2:** If a person took out a \$150,000 mortgage at 5% yearly interest, why would it be unwise to have monthly payments of \$500?

Now, even if we can define the sequence recursively, as in Exercise #1(d), it would be nice to have a formula that would calculate what we owed after a certain number of months explicitly. To do this, we must see a tricky, extended pattern.

***Exercise* #3:** Let's go back to our example of the \$200,000 mortgage at 4% yearly interest. Remember, we are paying off this mortgage with \$1,500 monthly payments (much of which are initially going to interest). Let's see if we can determine how much we still owe after *n*-payments. To make our work easier to follow (and more general), we will let $r=\frac{.04}{12}=.00\bar{3}$, $P=200{,}000$, and $m=1{,}500$ to stand for monthly rate (in decimal form, our principal, and our monthly payment).

(a) Explain what the following calculation represents.

$$P(1+r)-m$$

(b) What does the following calculation represent? Write it in expanded form, but leave the binomial $(1+r)$.

$$\left(P(1+r)-m\right)(1+r)-m$$

(c) Based on (a) and (b), continue this line of thinking to write expressions for the amount owed at the end of 3 months and 4 months.

(d) Based on (c), write an equation for how much would be owed after *n* months.

***Exercise* #4:** Now let's see the geometric series. In your answer to (d), you should have the following expression:

$$-m(1+r)^{n-1}-m(1+r)^{n-2}-m(1+r)^{n-3}-\cdots-m(1+r)^{2}-m(1+r)-m$$

Given that this is equivalent to: $-\left(m+m(1+r)+m(1+r)^{2}+m(1+r)^{3}+\cdots m(1+r)^{n-1}\right)$, find the sum of the geometric series inside of the parentheses. Think carefully about the number of terms in this expression.

***Exercise* #5:** Find the amount owed on this loan after 10 years or 120 payments (months).

Name: ___________________________________ Date: _________________

MORTGAGE PAYMENTS
COMMON CORE ALGEBRA II HOMEWORK

APPLICATIONS

1. Consider the mortgage loan of $150,000 at a nominal 6% yearly interest (so a monthly rate of 0.5%). Monthly payments of $1,000 are being made on this loan.

   (a) Determine how much is owed on this loan at the end of the first, second, third month and fourth months. Show the work that leads to your answers. Evaluate all expressions.

   One Month: Two Months:

   Three Months: Four Months:

   (b) The amounts that are owed at the end of each month form a sequence that can be defined **recursively**. Given that $a_1 = \$150,000$ represents the first amount owed, give a recursive rule based on what you did in (a) that shows how each successive amount owed depends on the previous one.

   (c) Using a geometric series approach (i.e. the formula we developed in Exercises #3 and #4), determine how much is still owed after 5 years of payments. Show your work.

   (d) Will this loan be paid off after 20 years? What about 30? Provide evidence to support both answers.

When loan officers speak to people about taking out a loan for a certain principal, P, at a certain monthly rate, r, they always have to balance two quantities, the monthly payment, m, with the number of payments, n, it takes to pay off the loan. These two vary **inversely**. All of these quantities can be related by the formula:

$$m = \frac{P \cdot r}{1-(1+r)^{-n}}$$

This formula is derived by taking the formula we arrived at in Exercises #3 and #4 and setting what we owe equal to zero.

2. Calculate the monthly payment needed to pay off a $200,000 loan at 4% yearly interest over a 20 year period. Show your work and carefully evaluate the above formula for m.

3. Do the same calculation as in the previous exercise but now make the pay off period 30 years instead of 20. How much less is your monthly payment?

It is of interest to also be able to calculate the number of payments (and hence the pay off period) if you have a monthly payment in mind. But this is much more difficult given that you must solve for n.

4. Given the formula above:

(a) Show that $n = \dfrac{-\log\left(1-\dfrac{P \cdot r}{m}\right)}{\log(1+r)}$

(b) Using (a), determine the number of months it would take to pay off a $150,000 loan at a monthly 0.5% rate with $1,000 payments.